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Author:

C.J. van Duijn[1,2], Andro Mikelic[3], Mary F. Wheeler[4], and Thomas Wick[5]

1. Department of Earth Sciences, Utrecht University, P.O. Box 80021, 3508TA Utrecht, The Netherlands
2. Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
3. Univ Lyon, Universite Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France
4. The Institute for Computational Engineering and Sciences, The University of Texas at Austin, Austin, Texas 78712, USA
5. Institute of Applied Mathematics, Leibniz Universitat Hannover Welfengarten 1, 30167 Hannover, Germany

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Thermoporoelasticity via homogenization I. Modeling and formal two-scale expansions*

C.J. van Duijn

Darcy Center, Eindhoven University of Technology, Department of Mechanical
Engineering and Utrecht University, Department of Earth Sciences,
The Netherlands

Andro Mikelić[†]
E-mail: `mikelic@univ-lyon1.fr`
Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208,
Institut Camille Jordan, 43 blvd. du 11 novembre 1918,
F-69622 Villeurbanne cedex, France

Mary F. Wheeler
The Institute for Computational Engineering and Sciences,
The University of Texas at Austin, Austin, Texas 78712, USA

Thomas Wick
Centre de Mathématiques Appliquées (CMAP), Ecole Polytechnique
91128 Palaiseau cedex, France
and
Institute of Applied Mathematics, Leibniz Universität Hannover
Welfengarten 1, 30167 Hannover, Germany

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Abstract

In this paper we derive a macroscopic model for thermoporoelasticity from the pore scale linearized fluid-structure and energy equations. We consider the continuum mechanics thermodynamically compatible pore scale equations corresponding to realistic rock mechanics parameters. They are upscaled using two-scale asymptotic expansions. For the upscaled equations a Lyapunov functional (a generalization of Biot's free energy) is constructed and the well-posedness

of the model is discussed. Possible applications to large time numerical simulations are pointed out.

1   Introduction

There are serious energy and environmental problems that require advanced mathematical and high fidelity numerical modeling of thermoporoelasticity processes in geomaterials. These include geological disposal of heat-emitting nuclear fuel (nuclear waste management), geological sequestration of green house gases in supercritical form, extractions of geothermal energy, mechanics of frozen soils, and enhanced oil recovery using steam injection and hot water flooding. Scientific issues that need to be addressed involve the mechanics of porous skeletons due to elastic, plastic and creep behaviour, fluid transport characteristic of the intact skeleton and alternations in the skeleton due to mechanical, fluid flow and heat transfer processes. Additional complexities arise due to anisotropy, heterogeneity of the porous media, geochemistry effects, and multiphase fluids. Efforts in the engineering design and management of these thermoporoelastic examples are difficult and costly. Predictive computational simulation may be the only means to account for the lack of complete characterization of the subsurface environment, the multiple scales of the various interacting processes, the large areal extent of saline aquifers in the case of carbon sequestration , and the need for long time predictions. Moreover, since constitutive functionals and parameters must be determined experimentally there is a need for coupling modeling at both the pore laboratory and field scales.

To further motivate the importance of thermoporoelastic studies, we refer to Rummelhoff [30] who stated that without carbon capture and storage, it is not realistic to meet the global climate target as defined in the Paris Agreement. A massive scale up of the number of carbon capture and storage (CCS) projects is needed, and collaboration and sharing of knowledge are essential to accelerating the development.

There are numerous isothermal CCS studies, including those carried out one of the authors of this paper. However, very limited modeling that includes thermal effects has been done to date. Our objective in this first paper is to focus on fundamental modeling aspects of thermoporoelasticity, using homogenization and formal two-scale expansions. Later work will focus on rigorous mathematical results, benchmark problems and on numerical simulation of nuclear waste and CCS examples.

The equations describing the coupling between mechanical effects and flow in porous media were first introduced in the pioneering papers of Biot: see for instance the collection of Biot's papers in Tolstoy [33]. In these works one roughly observes two regimes: *consolidation processes*, described by the quasi-static form of the equations, and *wave propagation*, described by their dynamic form.

To understand the origin of Biot's equations one distinguishes two approaches. One is thermodynamically based as explained in the work of Coussy et al ([11],[12],[13]) or Lewis and Schrefler [22].

The other approach – the so-called homogenization approach –considers Biot's equation as the effective (upscaled) form of the pore level (microscopic) equations.

Starting point in the homogenization approach are the continuum mechanics equations at the pore scale. Using the dimensionless pore size as the small parameter ε, the effective (upscaled) equations are found in the form of a power series in ε, with the coefficients being functions of the macroscopic variable x and the microscopic variable $y = x/\varepsilon$. Well-known references using such two-

scale expansions are Auriault and Sanchez-Palencia [5] and Burridge and Keller [9]. For a complete overview from an engineering perspective we refer to the monograph by Mei and Vernescu [25].

Rigorous mathematical results concerning the dynamic case are found in the monograph by Sanchez-Palencia [31]. Here the convergence of the Laplace transform in time of the displacements in the homogenization limit $\varepsilon \to 0$ was established. The complexity of such multi-scale problems motivated Nguetseng ([26], [27]) and Allaire [1] to introduce two-scale convergence techniques.

Rigorous results for the quasi-static regime, including interface conditions with an elastic body are given by Mikelić and Wheeler [24].

In this paper we are concerned with the problem of quasi-static thermal poroelasticity: coupling mechanics, flow and temperature in the regime where the acceleration terms are small.

The corresponding macroscopic equations being used in the engineering literature are often based on direct thermodynamic modeling. This approach extends Biot's pioneering work on the thermodynamics of deformable porous media (see for instance the collection [33]) and is described in the work of Coussy et al ([11], [12], [13]). It yields a global macroscopic model based on mixture theory, where the equations are posed separately for the fluid and solid phases. The theory also deals with nonlinear porous continua: a recapitulation of the governing equations is given in ([13], p. 58). Macroscopic constitutive equations for thermoelastic processes in a fluid-saturated porous medium, with small heat advection and small thermoelastic coupling, are presented in [32] .

A formal homogenization approach in the thermoelastic case is considered by Lee and Mei ([19], [20]). Using the equations at the pore level they consider the perturbation of an equilibrium state. This state is characterize by a hydrostatic equilibrium for the fluid and a static equilibrium for the initial temperature. The perturbation satisfies the Boussinesq approximation for the viscous thermal flow in the pores and the linear elasticity equations coupled to heat conduction in the solid skeleton. The resulting system is written in the Eulerian description. In this description the fluid-solid interface moves in time. Small displacements of order $\varepsilon\ell$, ℓ being the dimensional pore size, are assumed. This allows for an expansion of the interface conditions at the original static position. The imposed limitation on the microscopic displacement leads to a small upscaled displacement.

Comparing the homogenized constitutive laws in Lee and Mei to the results of Coussy et al, one observes that the poroelastic coefficients coincide but that the coefficients corresponding to the thermal effects deviate.

The goal of this paper is to revisit the derivation of the homogenized model, dropping the smallness conditions on the displacements and pressure (see equation (23) on page 129 of [19]). Moreover, we take into the account the effect of heat generation by deformation.

The pore scale equations are based on the assumptions:

i) The quasi-static deformation and heat flow in the elastic skeleton are described by the Navier linear elasticity system and the heat equation, while the stress-temperature tensor contribution is kept in the entropy. The equations are considered in the Lagrangian description.

ii) The fluid in the pores satisfies the Boussinesq approximation and inertia is disregarded.

Our aim is to derive an effective thermoporoelastic model in which the upscaled displacement can take values up to $1 - 5\%$ of the domain size and in which filtration rates have realistic values. In addition we keep the thermoelastic coupling term in the energy balance, which allows us to investigate its consequences. For instance, its presence is crucial in a derivation of a Lyapunov functional of the upscaled system.

We circumvent the difficulty caused by the moving fluid-solid interface by using the Lagrangian description for the equations of the elastic skeleton and the ALE (Arbitrary Lagrangian Eulerian)

([14],[18]) description for the fluid equations. The ALE description is such that the initial configuration of the fluid-solid interface remains unchanged for all times (e.g., [29], [36]).

The pore scale equations are introduced in Section 2 in detail and realistic data are used to write them in a dimensionless form. In Section 3 we present the upscaling of the dimensionless micro-scale model, as introduced in Section 2. The standard steps in the derivation of Biot's equations by the two-scale expansion are alternating with the expansion of the temperature equations. The summary of the upscaled equations is presented in Section 4. They take a form similar to the equations in ([13], page 75) and in [19]. However there are noticeable and non-trivial differences. In Coussy et al the equations are presented in a very general form, including large deviations. therefore it is difficult to compare terms in the equations and to quantify the (matricial) coefficients. In our approach one is able to compute all coefficients and matrices in terms of the microstructure. In Lee and Mei the term describing heat generation by the solid deformation is missing and some smallness assumptions are imposed. As a consequence, a quantitative comparison is virtually impossible.

In Section 4 we also give a Lyapunov functional for the upscaled equations. Its form is a generalization of the Biot free energy from the isothermal case. It has similarities with the free energy from [12], but it does not coincide. This stability property can be used in numerical algorithms when solving the equations for large time intervals. This is crucial in simulating problems of practical interest.

Section 5 is dedicated to the calculation and discussion of the properties of the homogenized coefficients and Section 6 contains the conclusions.

2 Basic model

In this section we derive the equations on the microscopic level as a preparation for the homogenization procedure. First we present in Section 2.1, for the ease of the reader, a short description of the thermoelastic equations as they appear in the literature (for instance in the classical monographs by Ziegler [39] or Gurtin et al [17]).

Then in Section 2.2 we set up the equations and interface conditions at the microscale. In Section 2.3 we discuss the scaling of the microscale problem, yielding the dimensionless formulation and the homogenization parameter ε.

2.1 Thermo-elastic equations

Let

$$e = e(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla^\tau \mathbf{u})$$

be the symmetric (infinitesimal) strain tensor. Throughout this paper we use the notation

$$A : B = \sum_{i,j} A_{ij} B_{ij} = tr(AB^\tau)$$

for any two matrices A and B.

The fundamental equations describing the displacement ($\mathbf{u}$) and the temperature (T) of an elastic body are derived from the following considerations.

Momentum balance equation:

$$- \operatorname{div} \sigma = \mathbf{b}, \tag{1}$$

where σ denotes the (Cauchy) stress tensor and $\mathbf{b}$ a given body force. Often this is the gravitational force $\mathbf{b} = \rho\mathbf{g}$ with ρ denoting the mass density of the material and $\mathbf{g}$ the gravity vector. In this balance equation we have neglected inertia effects $(\rho\frac{\partial^2\mathbf{u}}{\partial t^2})$.

Energy balance equation:

$$\frac{\partial E}{\partial t} = \sigma : e(\frac{\partial}{\partial t}\mathbf{u}) - \operatorname{div}\mathbf{q} + h, \tag{2}$$

$$\mathbf{q} = -\kappa\nabla T. \tag{3}$$

Here $E, \mathbf{q}, \kappa$ and h denote, respectively, the internal energy of the material, heat flux, thermal conductivity tensor and external heat source. The dissipation term $\sigma : e(\frac{\partial}{\partial t}\mathbf{u})$ will be removed from equations (2) and (3) when working in terms of the entropy (S).

Next we introduce the *free energy*

$$\psi = E - ST, \tag{4}$$

where S denotes the entropy.

In the theory of thermo-elasticity a general assumption is (see for instance [39]) that

$$\psi = \psi(e, T) \tag{5}$$

only, with

$$\sigma = \frac{\partial\psi}{\partial e} \quad (\sigma_{ij} = \frac{\partial\psi}{\partial e_{ij}}) \tag{6}$$

and

$$S = -\frac{\partial\psi}{\partial T}. \tag{7}$$

With this in mind, we first rewrite the energy balance (2). Differentiating (5) with respect to time (t) yields

$$\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial e} : e(\frac{\partial}{\partial t}\mathbf{u}) + \frac{\partial\psi}{\partial T}\frac{\partial T}{\partial t} = \sigma : e(\frac{\partial}{\partial t}\mathbf{u}) - S\frac{\partial T}{\partial t}. \tag{8}$$

Differentiating now (4) with respect to time gives

$$\frac{\partial\psi}{\partial t} = \frac{\partial E}{\partial t} - S\frac{\partial T}{\partial t} - T\frac{\partial S}{\partial t}. \tag{9}$$

Equations (8) and (9) imply

$$T\frac{\partial S}{\partial t} = \frac{\partial E}{\partial t} - \sigma : e(\frac{\partial}{\partial t}\mathbf{u}). \tag{10}$$

Using this in (2) gives the rewritten energy balance

$$T\frac{\partial S}{\partial t} = -\operatorname{div}\mathbf{q} + h. \tag{11}$$

The free energy is expanded at a reference state which is strain free $(e = 0)$ and has a reference temperature T_0. Up to second order this gives

$$\psi(e, T) = \psi_0 + \frac{\partial\psi}{\partial e}|_0 : e + \frac{\partial\psi}{\partial T}|_0(T - T_0) + \frac{1}{2}e : \frac{\partial^2\psi}{\partial e^2}|_0 e + (T - T_0)\frac{\partial^2\psi}{\partial e\partial T}|_0 : e$$

$$+ \frac{1}{2}\frac{\partial^2\psi}{\partial T^2}|_0(T - T_0)^2. \tag{12}$$

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Assuming that the reference state is stress free and has zero entropy (i.e. assuming $\sigma(0, T_0)$ and $S(0, T_0) = 0$, see (6)-(7)), and neglecting the constant ψ_0, expansion (12) reduces to

$$\psi(e, T) = \frac{1}{2} e : \frac{\partial^2 \psi}{\partial e^2}|_0 e + (T - T_0) \frac{\partial^2 \psi}{\partial e \partial T}|_0 : e + \frac{1}{2} \frac{\partial^2 \psi}{\partial T^2}|_0 (T - T_0)^2. \tag{13}$$

In classical linear thermo-elasticity one takes for (13) the specific expression

$$\psi(e, T) = \frac{1}{2} e : Ae + (T - T_0)\mathcal{M} : e - \frac{\rho c}{2T_0}(T - T_0)^2, \tag{14}$$

where A is the elasticity tensor (generalized Hooke's law) satisfying

$$A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij} \quad \text{and} \quad Ae : e > 0 \tag{15}$$

and where $\mathcal{M}$ is the symmetric stress-temperature tensor. Finally, ρ denotes the density of the material and c the specific heat.

In the case of an isotropic material one has

$$A_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{16}$$

implying

$$\sigma_{ij} = \lambda \delta_{ij} \sum_k e_{kk} + 2\mu e_{ij}. \tag{17}$$

Here λ and μ are the Lamé moduli. They are given by

$$\lambda = \frac{\Lambda \nu}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad \mu = \frac{\Lambda}{2(1 + \nu)}, \tag{18}$$

where Λ is Young's modulus and ν is Poisson's ratio. In the isotropic case, the stress-temperature tensor takes the form

$$\mathcal{M}_{ij} = -(3\lambda + 2\mu)\beta\delta_{ij}, \tag{19}$$

where β is the coefficient of thermal expansion.

With expression (14) one finds for the stress and entropy the expressions (the constitutive equations)

$$\sigma = \frac{\partial \psi}{\partial e} = Ae + (T - T_0)\mathcal{M} \tag{20}$$

and

$$S = -\frac{\partial \psi}{\partial T} = -\mathcal{M} : e + \frac{\rho c}{T_0}(T - T_0). \tag{21}$$

This implies for the momentum balance

$$-\operatorname{div}\Big(Ae(\mathbf{u}) + (T - T_0)\mathcal{M}\Big) = \mathbf{b}, \tag{22}$$

and for the energy balance, replacing $T\frac{\partial S}{\partial t}$ by $T_0\frac{\partial S}{\partial t}$ in (11),

$$\frac{\partial}{\partial t}\Big(\rho c(T - T_0) - T_0\mathcal{M} : e\Big) = \operatorname{div}(\kappa \nabla T) + h. \tag{23}$$

6

In this work we confine ourselves to isotropic solid materials. This makes the derivation of the upscaling procedure and its interpretation more transparent. For the non-isotropic case we refer to Remark 8.

The isotropic form of equations (22) and (23) reads

$$- \operatorname{div}\Big(Ae(\mathbf{u}) - (3\lambda + 2\mu)\beta(T - T_0)I \Big) = \mathbf{b} \tag{24}$$

and

$$\frac{\partial}{\partial t}\Big(\rho c(T - T_0) + (3\lambda + 2\mu)\beta T_0\, \operatorname{div}\mathbf{u} \Big) = \operatorname{div}(\kappa\nabla T) + h. \tag{25}$$

Introducing the dimensionless temperature

$$\vartheta = \frac{T - T_0}{\Delta T}, \quad \Delta T = \text{characteristic temperature difference}, \tag{26}$$

into equations (24) and (25) we arrive at

$$- \operatorname{div}\Big(Ae(\mathbf{u}) - (3\lambda + 2\mu)\beta\Delta T\vartheta I \Big) = \mathbf{b} \tag{27}$$

and

$$\frac{\partial}{\partial t}\Big(\rho c\vartheta + (3\lambda + 2\mu)\frac{\beta T_0}{\Delta T}\, \operatorname{div}\mathbf{u} \Big) = \operatorname{div}(\kappa\nabla\vartheta) + \frac{h}{\Delta T}. \tag{28}$$

Equations (27)-(28) are used to describe the displacement and temperature in the solid part of the porous structure.

2.2 Equations at the micro scale

Let us consider a geometrical structure as in Figure 1. This structure is repeated in all three directions to form a pavement Υ of the whole space. The characteristic size of the structure is denoted by ℓ (characteristic micro-scale length or pore size). The structure is decomposed as $\Upsilon = \Upsilon_f \cup \Upsilon_s \cup \Gamma_\Upsilon$, where Υ_f (the pore space) and Υ_s (the solid skeleton) are open and connected sets and where Γ_Υ is the interface between them.

Let $\tilde{\Omega}$ be a smooth domain in $\mathbb{R}^3$ with characteristic length L. Then we write $\tilde{\Omega} = \Omega_f^p \cup \Omega_s^p \cup \Gamma^p$, with

$\Omega_f^p = \tilde{\Omega} \cap \Upsilon_f$: Fluid part of $\tilde{\Omega}$, connected domain, filled with an incompressible fluid to which Stokes' equations apply.

$\Omega_s^p = \tilde{\Omega} \cap \Upsilon_s$: Solid part (skeleton), connected domain, behavior is described by the thermoelastic equations (27)-(28).

$\Gamma^p = \tilde{\Omega} \cap \Gamma_\Upsilon$: Fluid-solid interface with sufficient smoothness.

When describing the equations in $\tilde{\Omega}$ we use the Lagrangian approach in the solid part Ω_s^p and the so-called ALE (Arbitrary Lagrange-Eulerian) approach ([14], [18]) in the fluid part Ω_f^p. This means that the domains Ω_f^p and Ω_s^p (and the interface Γ^p) are fixed in time and that the fluid velocity $\mathbf{v}_f$ in Ω_f^p is written with respect to the velocity of the solid structure. To this end, we extend the
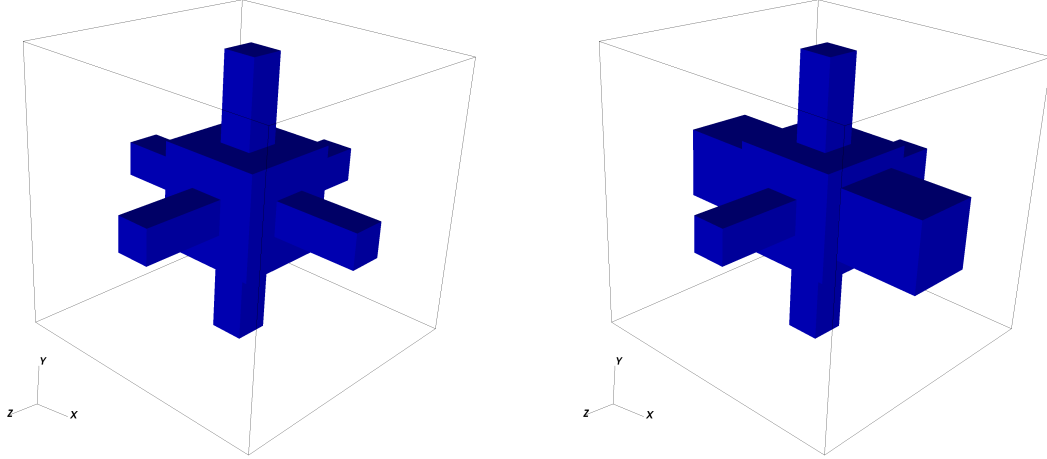
Figure 1: (left) Unit cell of isotropic pore network model, (right) unit cell of anisotropic pore network model. The fluid part is in blue. In Sections 3 - 5, this geometry corresponds to the notation $Y = (0,1)^3$ (unit cell), where Y_f is the fluid part and Y_s is the solid part.

displacement vector $\mathbf{u} = \mathbf{u}(x,t)$, with $x \in \Omega_s^p$, of the solid structure to the whole of $\tilde{\Omega}$, e.g. see Richter and Wick [29], and write the fluid velocity as

$$\mathbf{v}_f = \mathbf{w} + \frac{\partial}{\partial t}\mathbf{u} \quad \text{in } \Omega_f^p. \tag{29}$$

If the interface Γ^p acts as a no-slip boundary for the fluid we impose $\mathbf{v}_f = \frac{\partial}{\partial t}\mathbf{u}$ on Γ^p or, equivalently,

$$\mathbf{w}|_{\Gamma^p} = 0. \tag{30}$$

Note that in the ALE formulation one uses the fluid velocity $\mathbf{v}_f$ in the flow equations, but the relative velocity $\mathbf{w}$ in the energy transport equation, see for instance ([3], [29] and [35]).

2.2.1   Equations in the fluid domain Ω_f^p

In Ω_f^p we describe the fluid movement by Stokes' equations and the Boussinesq approximation, see ([15] , [16] , [28], [38]). Let the fluid stress σ^f be given by

$$\sigma^f = -pI + 2\eta(\vartheta)e(\mathbf{v}_f), \tag{31}$$

where p is the fluid pressure and $\eta(\vartheta)$ is the fluid viscosity in terms of the dimensionless temperature (26). Furthermore, let the fluid density ρ_f be given by

$$\rho_f = \rho_f^0(1 - \beta_f(T - T_0)) = \rho_f^0(1 - \beta_f \Delta T \vartheta) \tag{32}$$

where ρ_f^0 is the reference mass density of the fluid and β_f the coefficient of thermal expansion. Then from the fluid momentum balance

$$-\operatorname{div}\sigma^f = \rho_f \mathbf{g} = -\rho_f g \mathbf{e}_3, \tag{33}$$

where g is the gravity constant and $\mathbf{e}_3$ the unit vector pointing upwards in positive x_3 direction, we obtain

$$\nabla p - \text{div}(2\eta(\vartheta)e(\mathbf{v}_f)) = -\rho_f^0 g\mathbf{e}_3 + \rho_f^0 \beta_f \Delta T g\vartheta\mathbf{e}_3. \tag{34}$$

Absorbing the hydrostatic pressure into p by redefining

$$p := p + \rho_f^0 gx_3, \tag{35}$$

gives for the pressure and velocity of the fluid

$$\nabla p - \text{div}(2\eta(\vartheta)e(\mathbf{v}_f)) = \rho_f^0 \beta_f \Delta T g\vartheta\mathbf{e}_3. \tag{36}$$

Incompressibility requires

$$\text{div}\mathbf{v}_f = \text{div}(\mathbf{w} + \frac{\partial}{\partial t}\mathbf{u}) = 0. \tag{37}$$

Finally the fluid temperature satisfies (same notation for fluid and solid temperature)

$$\rho_f^0 c_f(\frac{\partial T}{\partial t} + \mathbf{w} \cdot \nabla T) = \text{div}(\kappa_f \nabla T) + h_f. \tag{38}$$

Here c_f is the specific heat of the fluid, κ_f the thermal conductivity tensor and h_f the external heat source. In terms of the dimensionless temperature (26) we obtain

$$\rho_f^0 c_f(\frac{\partial \vartheta}{\partial t} + \mathbf{w} \cdot \nabla \vartheta) = \text{div}(\kappa_f \nabla \vartheta) + \frac{h_f}{\Delta T}. \tag{39}$$

Remark 1. *In the case of large solid deformations, the flow equations would contain coefficients depending on the displacement $\mathbf{u}$ (e.g., see [29] and [35]). In the present ALE description this is disregarded.*

2.2.2   Equations in the solid domain Ω_s^p

They are given by equations (27) and (28). For completeness and because of adjusted notation ($\rho \to \rho_s$, $c \to c_s$, $\beta \to \beta_s$, etc) they are repeated here:

$$-\text{div}\sigma^s = -\text{div}(Ae(\mathbf{u}) - (3\lambda + 2\mu)\beta_s\Delta T\vartheta I) = \mathbf{b} \tag{40}$$

and

$$\frac{\partial}{\partial t}(\rho_s c_s\vartheta + (3\lambda + 2\mu)\frac{\beta_s T_0}{\Delta T} \text{div}\mathbf{u}) = \text{div}(\kappa_s \nabla \vartheta) + \frac{h_s}{\Delta T}. \tag{41}$$

2.2.3   Interface conditions at Γ^p

At Γ^p we impose (30) for the relative velocity $\mathbf{w}$. The balance of momentum gives

$$\sigma^f\mathbf{n} = \sigma^s\mathbf{n} \quad \text{at} \quad \Gamma^p, \tag{42}$$

where the fluid pressure is replaced by $p - \rho_f^0 gx_3$. For the temperature we impose

$$[\vartheta] = 0 \quad \text{and} \quad \kappa_f \nabla\vartheta \cdot \mathbf{n} = \kappa_s \nabla\vartheta \cdot \mathbf{n} \quad \text{at } \Gamma^p. \tag{43}$$

In conditions (42) and (43), $\mathbf{n}$ denotes the unit vector at Γ^p pointing in the direction of the fluid. In the remainder of this paper, this will always be the case.

2.3 Scaling of equations and conditions

To discuss the appropriate scaling we first introduce a number of characteristic quantities. Some are obvious, others are not. They will be chosen to obtain the correct balance in the equations and conditions. Dimensionless numbers that will appear are compared to ε, the ratio between the micro and the macro scale i.e.

$$\varepsilon = \frac{\ell}{L} \tag{44}$$

where ℓ is the pore scale length and L the macro reservoir/experimental scale. Typically for dolomites and layered clays $\ell = O(10^{-7}m)$, yielding a permeability from 1 to 10 millidarcy (md) (see for instance [7]). Under the scaling $x \to x/L$, $\tilde{\Omega}$, Ω_f^p, Ω_s^p and Γ^p become Ω, Ω_f^ε, Ω_s^ε and Γ^ε, respectively. Note that after the scaling, the characteristic dimensionless pore size becomes ε.

Let $p_c, u_c, \tau_c, v_c, \eta_c$ and Λ_c denote, respectively, a characteristic pressure, deformation, time, fluid velocity and elastic constant. With these values one defines dimensionless variables in the obvious way: $p := p/p_c \dots$ etc. Throughout this paper we take

$$\Lambda_c = \Lambda \quad \text{(Young's modulus)}$$

and

$$\eta_c = \eta(0) \quad \text{(viscosity at reference temperature } T_0\text{)}.$$

Let us start with condition (42). Introducing the characteristic values and using (18) gives

$$(-p_c(p - p_h x_3)I + 2\eta_c \frac{v_c}{L}\eta(\vartheta)e(\mathbf{w} + \frac{u_c}{v_c \tau_c}\frac{\partial}{\partial t}\mathbf{u})) \cdot \mathbf{n} =$$
$$(\Lambda \frac{u_c}{L}Ae(\mathbf{u}) - \frac{4 - 2\nu}{(1+\nu)(1-2\nu)}\Lambda\beta_s\Delta T\vartheta I) \cdot \mathbf{n} \tag{45}$$

at Γ^ε. Balancing the first term on the left and right gives

$$p_c = \Lambda \frac{u_c}{L} \quad \text{and thus} \quad p_h := \frac{\rho_f^0 gL}{p_c} = \frac{\rho_f^0 gL^2}{\Lambda u_c}. \tag{46}$$

where p_h is a measure for the dimensionless hydrostatic pressure. Leaving u_c and τ_c unspecified for the moment, we take for v_c the characteristic Darcy velocity

$$v_c = \frac{\ell^2}{\eta_c}\frac{p_c}{L} = \frac{\varepsilon^2 L p_c}{\eta_c} = \frac{\varepsilon^2 \Lambda u_c}{\eta_c} =: \frac{u_c}{\tau_T}, \tag{47}$$

where

$$\tau_T = \frac{\eta_c}{\varepsilon^2 \Lambda} \ [s]. \tag{48}$$

is the time scale related to the deformation of the medium in response to a change in pore pressure.

Dividing (45) by p_c from (46) gives

$$(-(p - p_h x_3)I + 2\varepsilon^2\eta(\varepsilon)e(\mathbf{w} + \frac{\tau_T}{\tau_c}\frac{\partial}{\partial t}\mathbf{u})) \cdot \mathbf{n} =$$
$$(Ae(\mathbf{u}) - K_S\vartheta I) \cdot \mathbf{n} \quad \text{at } \Gamma^\varepsilon. \tag{49}$$

Here

$$K_S = \frac{4 - 2\nu}{(1+\nu)(1-2\nu)}\beta_s\Delta T\frac{L}{u_c}. \tag{50}$$

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Using the same scaling in the momentum balance equation for the solid structure (40) gives

$$- \operatorname{div}(Ae(\mathbf{u}) - K_S \vartheta I) = \frac{L^2}{\Lambda u_c} \mathbf{b} =: -K_g \mathbf{e}_3 \quad \text{in } \Omega_s^\varepsilon, \tag{51}$$

where

$$K_g = \frac{L^2 \rho_s g}{\Lambda u_c} = \frac{\rho_s}{\rho_f^0} p_h. \tag{52}$$

Next we consider Stokes' equations (35). Applying the scaling with (46), (47) and (48) gives

$$\nabla p - 2\varepsilon^2 \operatorname{div}(\eta(\vartheta)e(\mathbf{w} + \frac{\tau_T}{\tau_c} \frac{\partial}{\partial t} \mathbf{u})) = K_B \vartheta \mathbf{e}_3 \quad \text{in } \Omega_f^\varepsilon, \tag{53}$$

where

$$K_B = K_g \frac{\rho_f^0}{\rho_s} \beta_f \Delta T. \tag{54}$$

The incompressibility condition becomes

$$\operatorname{div}(\mathbf{w} + \frac{\tau_T}{\tau_c} \frac{\partial}{\partial t} \mathbf{u}) = 0 \quad \text{in } \Omega_f^\varepsilon. \tag{55}$$

It remains to scale the temperature equations (39) and (41). These equations contain the heat capacities $\rho_f^0 c_f$ and $\rho_s c_s$, both considered constant here. It is convenient at this point to introduce the average heat capacity

$$\tilde{\rho} c = \Phi \rho_f^0 c_f + (1 - \Phi)\rho_s c_s, \tag{56}$$

where $\Phi = |\Omega_f^\varepsilon|/|\Omega|$ denotes the porosity of the porous structure. The scaled equation for the fluid temperature becomes

$$C_f \frac{\partial \vartheta}{\partial t} + \frac{v_c \tau_c}{L} C_f \mathbf{w} \cdot \nabla \vartheta = \mathcal{D}_f \Delta \vartheta + \tilde{h}_f \quad \text{in } \Omega_f^\varepsilon, \tag{57}$$

where

$$\begin{cases} C_f = \frac{\rho_f^0 c_f}{\tilde{\rho} c}, \\ \mathcal{D}_f = \frac{\kappa_f \tau_c}{\tilde{\rho} c L^2} = D_f \frac{\tau_c}{L^2} \\ \tilde{h}_f = \frac{\tau_c}{\tilde{\rho} c \Delta T} h_f. \end{cases} \tag{58}$$

Here $D_f = \kappa_f / \tilde{\rho} c$ denotes the dimensional thermal diffusivity of the fluid. Note that Péclet's number in (57) is

$$\text{Pe} := \frac{v_c \tau_c}{L \mathcal{D}_f} C_f = \frac{\rho_f^0 c_f v_c L}{\kappa_f} \tag{59}$$

The equation for the temperature in the solid structure reads

$$\frac{\partial}{\partial t}(C_s \vartheta + K_T \operatorname{div} \mathbf{u}) = \mathcal{D}_s \Delta \vartheta + \tilde{h}_s \quad \text{in } \Omega_s^\varepsilon, \tag{60}$$

with

$$\begin{cases} C_s = \frac{\rho_s c_s}{\tilde{\rho} c}, \\ K_T = \frac{4 - 2\nu}{(1 + \nu)(1 - 2\nu)} \frac{\Lambda \beta_s T_0}{\tilde{\rho} c \Delta T} \frac{u_c}{L}, \\ \mathcal{D}_s = \frac{\kappa_s \tau_c}{\tilde{\rho} c L^2} = D_s \frac{\tau_c}{L^2} \\ \tilde{h}_s = \frac{\tau_c}{\tilde{\rho} c \Delta T} h_s. \end{cases} \tag{61}$$

Here $D_s = \kappa_s / \tilde{\rho} c$ denotes the dimensional thermal diffusivity of the solid. Note that

$$\Phi C_f + (1 - \Phi) C_s = 1.$$

The temperature flux condition (43) becomes

$$\mathcal{D}_f \nabla \vartheta \cdot \mathbf{n} = \mathcal{D}_s \nabla \vartheta \cdot \mathbf{n} \quad \text{at } \Gamma^\varepsilon.$$

Remark 2. *When writing (57) and (60) we silently replaced the heat conduction tensors by positive constants κ_f and κ_s.*

Remark 3. *In the quasi-static setting we are mainly interested in a filtration dominated regime. Then intuitively one would think of a decoupling, where first the filtration through a rigid porous structure is calculated and then the deformation of the solid skeleton is added a posteriori. In such a setting the characteristic Darcy velocity is given by (47), with p_c being the pressure drop. It originates from either the gravity forces, with $p_c = \rho_f^0 g L$ and $u_c = L^2 \rho_f^0 g / \Lambda$ or from the applied pressure at the outer boundary. The linear elasticity equations hold if the relative displacement is sufficiently small, say $u_c / L = \tau_T / \tau_c = p_c / \Lambda < O(10^{-2})$.*

Discussion on the choice of scales

First we recall in Table 1 some dimensional solid and fluid characteristics (order of magnitude). These values are taken from the literature, (e.g. [19], [23], [40]).

QUANTITY	VALUE FOR ROCKS	VALUE FOR CLAYS
Young's modulus	$\Lambda = 10^{10}$ Pa	$\Lambda = 10^7$ Pa
Solid thermal expansion coefficient	$\beta_s = 0.5 \cdot 10^{-5}/^0$K	$\beta_s = 10^{-6}/^0$K
Fluid thermal expansion coefficient	$\beta_f = 10^{-4}/^0$K	$\beta_f = 10^{-4}/^0$K
Solid thermal diffusivity	$D_s = 10^{-6} m^2/s$	$D_s = 10^{-6} m^2/s$
Fluid thermal diffusivity	$D_f = 10^{-7} m^2/\text{ s}$	$D_f = 10^{-7} m^2/\text{ s}$
Rock mass density	$\rho_s = 2.5 \cdot 10^3 kg/m^3$	$\rho_s = 2.5 \cdot 10^3 kg/m^3$
Fluid mass density	$\rho_f^0 = 10^3 kg/m^3$	$\rho_f^0 = 10^3 kg/m^3$
Water viscosity	$\eta_c = 10^{-3}$ Pa s	$\eta_c = 10^{-3}$ Pa s
Macroscopic length	$L = 1$ m	$L = 1$ m
Pore scale size	$\ell = 10^{-7}$ m	$\ell = 10^{-7}$ m
Small parameter	$\varepsilon = \ell/L = 10^{-7}$	$\varepsilon = \ell/L = 10^{-7}$
Average heat capacity	$\tilde{\rho} c = 10^6$ J/ (m$^{3\ 0}$ K)	$\tilde{\rho} c = 10^6$ J/ (m$^{3\ 0}$ K)

Table 1: *Order of the magnitude of data*

Motivation

We take $L = 1$m as the macro (Darcy) scale and $\ell = 10^{-7}$ m as the pore scale. Note that this leads to a permeability of $O(10^{-14} m^2)$ which is representative for various rock and soil types. For rock type material we choose a characteristic pressure $p_c = 10^7$ Pa, giving (from (46)) for $u_c = O(10^{-3} m)$. For clay type material we choose $p_c = 10^5$ Pa, yielding $u_c = O(10^{-2} m)$. With these values we have a clear separation of scales in the sense that

$$\ell \ll u_c \ll L.$$

Further we choose $\tau_c = L/v_c$ in (57) and (58). This gives for Péclet's number (59)

$$\text{Pe} = \frac{\rho_f^0 c_f}{\kappa_f} \frac{L^2}{\tau_c}.$$

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Using (47) we note that

$$\frac{\tau_T}{\tau_c} = \frac{u_c}{L}.$$

Remark 4. *If we compare our assumptions to [19], it turns out that these authors assumed $p_c/\Lambda = O(\varepsilon^2)$, see their formula (23) on page 129. Such a strong assumption is not necessary in our approach.*

With the choices and values from Table 1 we obtain for the other characteristic quantities and dimensionless numbers the values (with $\Delta T = 50\ ^0K$).

SYMBOL	*QUANTITY*	*ROCK VALUE*	*CLAY VALUE*
$\tau_T = \eta_C/(\Lambda\varepsilon^2)$	characteristic deformation time	10 s	10^4 s
$p_h = \rho_f^0 gL/p_c$	ratio hydrostatic vs pore pressures	10^{-3}	10^{-1}
$v_c = u_c/\tau_T$	velocity of the structure deformation	10^{-4} m/s	10^{-6} m/s
$K_S = \beta_s\Delta TL/u_c$	solid expansion coefficient	$5\cdot 10^{-2}$	$5\cdot 10^{-3}$
$K_g = \rho_s L^2 g/(\Lambda u_c)$	solid gravitational coefficient	$2.5\cdot 10^{-3}$	$2.5\cdot 10^{-1}$
$K_B = K_g\rho_f^0\beta_f\Delta T/\rho_s$	buoyancy coefficient	$5\cdot 10^{-6}$	$5\cdot 10^{-4}$
τ_c	transport time	10^4 s	10^6 s
$\mathcal{D}_f = D_f\tau_c/L^2$	scaled fluid diffusivity	10^{-3}	10^{-1}
$\mathcal{D}_s = D_s\tau_c/L^2$	scaled solid diffusivity	10^{-2}	1
$K_T = \Lambda\beta_s T_0 u_c/(\tilde\rho c\Delta TL)$	thermoelastic coupling parameter	$3\cdot 10^{-4}$	$6\cdot 10^{-7}$

Table 2: *Data description*

Remark 5. *We observe that for both type of materials the factor K_T is very small and can be disregarded for any practical or computational purposes. Note that one can write*

$$K_T = \frac{p_c\beta_s T_0}{\tilde\rho c\Delta T}.$$

Hence for considerably larger pressures this factor could gain importance. Therefore we keep it in the two-scale expansions. The smallness of the thermoelastic coupling parameter K_T was already noticed in [23].

Remark 6. *Let $\mathbf{u}^\varepsilon$ denote the dimensionless displacement. Then*

$$\mathbf{u}^\varepsilon = \mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \dots$$

Hence the actual displacement can be written as

$$u_c\mathbf{u}^\varepsilon = \underbrace{u_c\mathbf{u}^0}_{macro\ displacement} + \underbrace{u_c\frac{\ell}{L}\mathbf{u}^1}_{displacement\ in\ the\ pore} + \dots$$

Observe that the pore displacement is much smaller than the macroscopic displacement.

3 Upscaling through two-scale expansions

3.1 Geometry

We start by some comments on the geometry:

- Porous media structures met in nature are hard to describe. In order to model their shape it is necessary to pose some hypotheses. In general it is supposed that there are two connected phases: a fluid phase and a solid phase, the latter being either rigid or deformable. Furthermore, the porous medium is supposed to be heterogeneous at the microscopic (pore) level but statistically homogeneous at the macroscopic level.

- The characteristic length of the scaled pores is ε. Since the theory for the true velocities, pressures and other quantities is too complicated, one prefers working with the averaged quantities over characteristic volumes being of order ε^3.

A representative example of the geometry is the *periodic* porous medium with connected fluid and solid phases. It is obtained by a periodic arrangement of the pores. After dividing by the characteristic length L, the porous medium becomes a box $\Omega = (0, R)^3$ in $\mathbb{R}^3$, $R > 0$. The description goes along the following lines:

First define the geometrical structure inside the unit cell $Y = (0, 1)^3$. Let Y_s (the solid part) be an open subset of Y and $Y_f = Y \backslash \bar{Y}_s$ (the fluid part). The liquid/solid interface is $S = \partial Y_s \setminus \partial Y$. It is assumed, see for instance the unit cells in Figure 1, that Y_s is such that (i) if we repeat Y_s all over $\mathbb{R}^3$ we obtain a periodic solid skeleton E_S which is connected and (ii) the complement of $\bar{E}_S$ in $\mathbb{R}^3$ is the pore space E_F which is connected as well.

Now $\Omega = (0, R)^3$ is covered with a regular mesh of size ε, each cell being a cube $Y_i^\varepsilon = \varepsilon(Y + i)$, with $1 \leq i \leq N(\varepsilon) = |\Omega|\varepsilon^{-3}[1 + 0(1)]$. We define $Y_{s_i}^\varepsilon = \varepsilon(Y_s + i)$ and $Y_{f_i}^\varepsilon = \varepsilon(Y_f + i)$. For sufficiently small $\varepsilon > 0$ we consider $T_\varepsilon = \{k \in \mathbb{Z}^3 | Y_{s_k}^\varepsilon \subset \Omega\}$ and define

$$\Omega_s^\varepsilon = \bigcup_{k \in T_\varepsilon} Y_{s_k}^\varepsilon, \quad \Gamma^\varepsilon = \partial\Omega_s^\varepsilon \setminus \partial\Omega, \quad \Omega_f^\varepsilon = \Omega \setminus \overline{\Omega}_s^\varepsilon.$$

The domains Ω_s^ε and Ω_f^ε represent, respectively, the solid and fluid parts of a porous medium Ω.

For simplicity, we will suppose that L/ε is an *integer*. Note that $|Y_f| = \Phi$ is the porosity of the reference configuration.

Remark 7. *The sketched geometry is the reference configuration in the Lagrangian description of the deformation of the elasticity part.*

3.2 Pore level equations

Recalling the equations from Subsection 2.3, we consider the dimensionless system of partial differential equations for $t \in (0, \mathcal{T})$, $\mathcal{T} > 0$:

$$- \operatorname{div}\sigma^{f,\varepsilon} = (K_B \vartheta^\varepsilon - p_h)\mathbf{e}_3 \quad \text{in} \quad \Omega_f^\varepsilon, \tag{62}$$

$$\operatorname{div}(\mathbf{w}^\varepsilon + \frac{\tau_T}{\tau_c}\frac{\partial \mathbf{u}^\varepsilon}{\partial t}) = 0 \quad \text{in} \quad \Omega_f^\varepsilon, \tag{63}$$

$$\operatorname{div}\sigma^{s,\varepsilon} - K_g \mathbf{e}_3 = 0 \quad \text{in} \quad \Omega_s^\varepsilon, \tag{64}$$

$$\mathbf{w}^\varepsilon = 0 \quad \text{on} \quad \Gamma^\varepsilon, \tag{65}$$

$$\sigma^{f,\varepsilon} = -(p^\varepsilon - p_h x_3)I + 2\varepsilon^2 \eta(\vartheta^\varepsilon)e(\mathbf{w}^\varepsilon + \frac{\tau_T}{\tau_c}\frac{\partial \mathbf{u}^\varepsilon}{\partial t}) \quad \text{in} \quad \Omega_f^\varepsilon, \tag{66}$$

$$\sigma^{s,\varepsilon} = Ae(\mathbf{u}^\varepsilon) - K_S \vartheta^\varepsilon I \quad \text{in} \quad \Omega_s^\varepsilon, \tag{67}$$

$$\sigma^{s,\varepsilon}\mathbf{n} = \sigma^{f,\varepsilon}\mathbf{n} \quad \text{on} \quad \Gamma^\varepsilon, \tag{68}$$

$$C_f\left(\frac{\partial \vartheta^\varepsilon}{\partial t} + \mathbf{w}^\varepsilon \cdot \nabla \vartheta^\varepsilon\right) = \mathcal{D}_f \Delta \vartheta^\varepsilon + \tilde{h}_f \quad \text{in} \quad \Omega_f^\varepsilon, \tag{69}$$

$$\frac{\partial}{\partial t}\left(C_s \vartheta^\varepsilon + K_T \operatorname{div} \mathbf{u}^\varepsilon\right) = \mathcal{D}_s \Delta \vartheta^\varepsilon + \tilde{h}_s \quad \text{in} \quad \Omega_s^\varepsilon, \tag{70}$$

$$\mathcal{D}_f\mathbf{n} \cdot \nabla \vartheta^\varepsilon = \mathcal{D}_s\mathbf{n} \cdot \nabla \vartheta^\varepsilon \quad \text{and} \quad [\vartheta^\varepsilon] = 0 \quad \text{on } \Gamma^\varepsilon. \tag{71}$$

Note that τ_c is the heat convection time (transport time) and $\tau_T/\tau_c = u_c/L$.

3.3 The two-scale asymptotic expansion

Due to the geometrical structure, our problem has two natural scales: the macro-scale x (the core scale) and the micro-scale $y = \dfrac{x}{\varepsilon}$. We search for $\{\mathbf{u}^\varepsilon, \vartheta^\varepsilon, \mathbf{w}^\varepsilon, p^\varepsilon\}$ in the following two-scale expansion form

$$\mathbf{u}^\varepsilon(x,t) = \mathbf{u}^0(x,y,t) + \varepsilon\mathbf{u}^1(x,y,t) + \varepsilon^2\mathbf{u}^2(x,y,t) + \dots, \tag{72}$$

$$\vartheta^\varepsilon(x,t) = \vartheta^0(x,y,t) + \varepsilon\vartheta^1(x,y,t) + \varepsilon^2\vartheta^2(x,y,t) + \dots, \tag{73}$$

$$\mathbf{w}^\varepsilon(x,t) = \mathbf{w}^0(x,y,t) + \varepsilon\mathbf{w}^1(x,y,t) + \varepsilon^2\mathbf{w}^2(x,y,t) + \dots, \tag{74}$$

$$p^\varepsilon(x,t) = p^0(x,y,t) + \varepsilon p^1(x,y,t) + \varepsilon^2 p^2(x,y,t) + \dots, \tag{75}$$

$\boxed{\text{STEP 1}}$

First we show that the effective displacement does not oscillate. To this end we substitute the expansion into equations (64) and (67) to get:

$$K_g \mathbf{e}_3 = \varepsilon^{-2} \operatorname{div}_y(Ae_y(\mathbf{u}^0)) + \varepsilon^{-1}\big(\operatorname{div}_y(Ae_y(\mathbf{u}^1) - K_S\vartheta^0 I + Ae_x(\mathbf{u}^0)) + \operatorname{div}_x(Ae_y(\mathbf{u}^0))\big) +$$
$$\varepsilon^0\big(\operatorname{div}_y(Ae_y(\mathbf{u}^2) + Ae_x(\mathbf{u}^1) - K_S\vartheta^1 I) + \operatorname{div}_x(Ae_y(\mathbf{u}^1) - K_S\vartheta^0 I +$$
$$Ae_x(\mathbf{u}^0)) + O(\varepsilon) \quad \text{in } Y_s \times \Omega \times (0, \mathcal{T}). \tag{76}$$

Expanding interface condition (68) gives

$$\sigma^{s,\varepsilon} \cdot \mathbf{n}|_{\Gamma^\varepsilon} = \big(\varepsilon^{-1}Ae_y(\mathbf{u}^0) \cdot \mathbf{n} + (Ae_y(\mathbf{u}^1) - K_S\vartheta^0 I + Ae_x(\mathbf{u}^0)) \cdot \mathbf{n} +$$
$$\varepsilon(Ae_y(\mathbf{u}^2) - K_S\vartheta^1 I + Ae_x(\mathbf{u}^1)) \cdot \mathbf{n})|_{\Gamma^\varepsilon} = \sigma^{f,\varepsilon} \cdot \mathbf{n}|_{\Gamma^\varepsilon} = O(1). \tag{77}$$

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At the order $O(\varepsilon^{-2})$, for any $(x,t) \in \Omega \times (0, \mathcal{T})$ we have

$$
\begin{cases}
\operatorname{div}_y\big(Ae_y(\mathbf{u}^0)\big) = 0 & \text{in} \quad Y_s, \\
Ae_y(\mathbf{u}^0) \cdot \mathbf{n} = 0 \text{ on } S, \ \mathbf{u}^0 \text{ is 1-periodic.}
\end{cases}
\tag{78}
$$

Since the solid structure is connected, as we are in $\mathbb{R}^3$, the only possible rigid motion is a constant translation with respect to y. Hence we have

$$
\mathbf{u}^0(x, y, t) = \mathbf{u}^0(x, t) \ \text{ in } \ Y_s \times \Omega \times (0, \mathcal{T}).
\tag{79}
$$

Therefore the displacement of the solid structure doesn't depend on the fast variable at order zero. Its extension to the pore space is also non-oscillatory.

$\boxed{\text{STEP 2}}$

Next we show that the effective pressure does not oscillate in the pores and write the Stokes system for $\mathbf{w}^0$ in the pores. Insert now the two-scale expansions into equations (62), (63) and (65)-(66). It yields

$$
\sigma^{f,\varepsilon} = -(p^0 - p_h x_3)I + \varepsilon\Big(-p^1 + 2\eta(\vartheta^0)e_y(\mathbf{w}^0 + \frac{\tau_T}{\tau_c}\frac{\partial \mathbf{u}^0}{\partial t})\Big) + O(\varepsilon^2) \ \text{ in } \ Y_f \times \Omega \times (0, \mathcal{T}),
\tag{80}
$$

$$
\varepsilon^{-1}\nabla_y p^0 + \varepsilon^0\big(\nabla_y p^1 + \nabla_x(p^0 - p_h x_3) - 2\operatorname{div}_y(\eta(\vartheta^0)e_y(\mathbf{w}^0)) - (K_B\vartheta^0 - p_h)\mathbf{e}_3\big) = O(\varepsilon)
\tag{81}
$$

in $Y_f \times \Omega \times (0, \mathcal{T})$. At order $O(\varepsilon^{-1})$ we get

$$
p^0(x, t, y) = p^0(x, t) \quad \text{in} \quad Y_f \times \Omega \times (0, \mathcal{T}).
\tag{82}
$$

Extending the pressure by $p^0 = 0$ in $\bar{Y}_s \times \Omega \times (0, \mathcal{T})$ we have

$$
p^0(x, t, y) = \chi_{Y_f}(y)p^0(x, t),
$$

where χ_{Y_f} denotes the characteristic function of the set Y_f.

Furthermore, the mass conservation equation becomes

$$
\varepsilon^{-1}\operatorname{div}_y\mathbf{w}^0 + \operatorname{div}_x\big(\mathbf{w}^0 + \frac{\tau_T}{\tau_c}\frac{\partial \mathbf{u}^0}{\partial t}\big) + \operatorname{div}_y\big(\mathbf{w}^1 + \frac{\tau_T}{\tau_c}\frac{\partial \mathbf{u}^1}{\partial t}\big) + O(\varepsilon) = 0
\tag{83}
$$

and at order $O(\varepsilon^0)$ we have the following Stokes problem

$$
\nabla_y p^1 + \nabla_x(p^0 - p_h x_3) - 2\operatorname{div}_y(\eta(\vartheta^0)e_y(\mathbf{w}^0)) = (K_B\vartheta^0 - p_h)\mathbf{e}_3 \ \text{ in } \ Y_f \times \Omega \times (0, \mathcal{T}),
\tag{84}
$$

$$
\operatorname{div}_y\mathbf{w}^0 = 0 \ \text{ in } \ Y_f \times \Omega \times (0, \mathcal{T}),
\tag{85}
$$

$$
\mathbf{w}^0 = 0 \ \text{ on } \ S \times \Omega \times (0, \mathcal{T}),
\tag{86}
$$

$$
\{\mathbf{w}^0, p^1\} \quad \text{are } 1 - \text{periodic in } \ y.
\tag{87}
$$

Comparison between (77) and (80) yields

$$
\varepsilon^{-1}Ae_y(\mathbf{u}^0) \cdot \mathbf{n} + (Ae_y(\mathbf{u}^1) - K_S\vartheta^0 I + Ae_x(\mathbf{u}^0) + (p^0 - p_h x_3))I) \cdot \mathbf{n} + \varepsilon(Ae_y(\mathbf{u}^2) - K_S\vartheta^1 I +
$$
$$
Ae_x(\mathbf{u}^1) + p^1 I - 2\eta(\vartheta^0)e_y(\mathbf{w}^0)) \cdot \mathbf{n} + O(\varepsilon^2) = 0
$$
$$
\text{on} \quad S \times \Omega \times (0, \mathcal{T}).
\tag{88}
$$

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Now we are able to go back to the expansion for the elasticity system and study the problem for $\mathbf{u}^1$.

STEP 3

Using (76) and (88) yields at order $O(\varepsilon^{-1})$

$$
\begin{cases}
\operatorname{div}_y\big(A(e_y(\mathbf{u}^1) + e_x(\mathbf{u}^0) - K_S\vartheta^0 I)\big) + \operatorname{div}_x(A\underbrace{e_y(\mathbf{u}^0)}_{=0}) = 0 \quad \text{in} \quad Y_s \times \Omega \times (0,\mathcal{T}), \\
\big(A(e_y(\mathbf{u}^1) + e_x(\mathbf{u}^0)) - K_S\vartheta^0 I\big) \cdot \mathbf{n} = -(p^0 - p_h x_3)\mathbf{n} \text{ on } S \times \Omega \times (0,\mathcal{T}), \\
\mathbf{u}^1 \quad \text{is 1-periodic.}
\end{cases}
\tag{89}
$$

By (82), p^0 does not depend on y on S and $\int_S p^0\mathbf{n}\,dS_y = 0$. Therefore, by the Fredholm alternative[1],

problem (89) has a solution. The solution is unique if we impose $\int_{Y_s} \mathbf{u}^1\,dy = 0$.

STEP 4

At this point we need information concerning ϑ^0. Inserting the two-scale expansions into equations (69)-(71) gives

$$
-\varepsilon^{-2}\mathcal{D}_f\Delta_y\vartheta^0 + \varepsilon^{-1}\big(C_f\mathbf{w}^0 \cdot \nabla_y\vartheta^0 - \mathcal{D}_f\Delta_y\vartheta^1 - 2\mathcal{D}_f\operatorname{div}_x\nabla_y\vartheta^0\big) + \varepsilon^0\big(C_f\frac{\partial\vartheta^0}{\partial t} +
$$
$$
C_f(\mathbf{w}^0 \cdot \nabla_y\vartheta^1 + \mathbf{w}^0 \cdot \nabla_x\vartheta^0 + \mathbf{w}^1 \cdot \nabla_y\vartheta^0) - \mathcal{D}_f\Delta_y\vartheta^2 - 2\mathcal{D}_f\operatorname{div}_x\nabla_y\vartheta^1 - \mathcal{D}_f\Delta_x\vartheta^0 - \tilde{h}_f\big) =
$$
$$
O(\varepsilon) \quad \text{in} \quad Y_f \times \Omega \times (0,\mathcal{T}),
\tag{90}
$$

$$
-\varepsilon^{-2}\mathcal{D}_s\Delta_y\vartheta^0 + \varepsilon^{-1}\big(K_T\operatorname{div}_y\mathbf{u}^0 - \mathcal{D}_s\Delta_y\vartheta^1 - 2\mathcal{D}_s\operatorname{div}_x\nabla_y\vartheta^0\big) + \varepsilon^0\big(C_s\frac{\partial\vartheta^0}{\partial t} + K_T(\operatorname{div}_y\mathbf{u}^1 +
$$
$$
\operatorname{div}_x\mathbf{u}^0) - \mathcal{D}_s\Delta_y\vartheta^2 - 2\mathcal{D}_s\operatorname{div}_x\nabla_y\vartheta^1 - \mathcal{D}_s\Delta_x\vartheta^0 - \tilde{h}_s\big) = O(\varepsilon) \quad \text{in} \quad Y_s \times \Omega \times (0,\mathcal{T}),
\tag{91}
$$

$$
\varepsilon^{-1}\big(\mathcal{D}_f\nabla_y\vartheta^0 \cdot \mathbf{n} - \mathcal{D}_s\nabla_y\vartheta^0 \cdot \mathbf{n}\big) + \mathcal{D}_f(\nabla_y\vartheta^1 + \nabla_x\vartheta^0) \cdot \mathbf{n} - \mathcal{D}_s(\nabla_y\vartheta^1 + \nabla_x\vartheta^0) \cdot \mathbf{n} +
$$
$$
\varepsilon\big(\mathcal{D}_f(\nabla_y\vartheta^2 + \nabla_x\vartheta^1) \cdot \mathbf{n} - \mathcal{D}_s(\nabla_y\vartheta^2 + \nabla_x\vartheta^1) \cdot \mathbf{n}\big) = O(\varepsilon^2) \quad \text{on} \quad S \times \Omega \times (0,\mathcal{T}).
\tag{92}
$$

We recall that ϑ^j is continuous across $S \times \Omega \times (0,\mathcal{T})$ for every j. The leading order terms in the two-scale expansions (90)-(92) yield the following problem at order $O(\varepsilon^{-2})$:

$$
-\mathcal{D}_f\Delta_y\vartheta^0 = 0 \quad \text{in} \quad Y_f,
\tag{93}
$$
$$
-\mathcal{D}_s\Delta_y\vartheta^0 = 0 \quad \text{in} \quad Y_s,
\tag{94}
$$
$$
\mathcal{D}_f\nabla_y\vartheta^0 \cdot \mathbf{n} = \mathcal{D}_s\nabla_y\vartheta^0 \cdot \mathbf{n} \quad \text{and} \quad [\vartheta^0] = 0 \quad \text{on} \quad S,
\tag{95}
$$
$$
\vartheta^0 \quad \text{is} \quad 1 - \text{periodic},
\tag{96}
$$

for every $(x,t) \in \Omega \times (0,\mathcal{T})$.

---

[1]Comment for the non-mathematical reader: Fredholm's alternative gives a necessary and sufficient criteria for solvability of an equation, in the critical situation when we are in a spectrum. For linear algebraic system $\mathcal{A}\mathbf{x} = \mathbf{b}$, it says that if 0 is an eigenvalue of the matrix $\mathcal{A}$, the system has a solution if and only if $\mathbf{b}$ is orthogonal to the eigenvectors of $\mathcal{A}$ that correspond to the eigenvalue 0. In all examples considered here 0 is an eigenvalue, with the eigenfunction being either a constant or a constant vector. Therefore the corresponding boundary value problem in the y-variable admits a solution if and only if the volume average of the right hand side with respect with respect to y is equal to the interface average of the flux. We refer to the textbooks as [37], for the Fredholm theory for the partial differential equations.

Only possible solution for problem (93)-(96) is a constant in y. Hence $\vartheta^0 = \vartheta^0(x,t)$.

The next order terms in the expansions (90)-(92) yield the following problem at order $O(\varepsilon^{-1})$:

$$-\mathcal{D}_f \operatorname{div}_y(\nabla_y \vartheta^1 + \nabla_x \vartheta^0) + C_f \mathbf{w}^0 \cdot \underbrace{\nabla_y \vartheta^0}_{=0} - \mathcal{D}_f \operatorname{div}_x \underbrace{\nabla_y \vartheta^0}_{=0} = 0 \quad \text{in} \quad Y_f, \tag{97}$$

$$K_T \frac{\partial}{\partial t} \underbrace{\operatorname{div}_y \mathbf{u}^0}_{=0} - \mathcal{D}_s \operatorname{div}_y(\nabla_y \vartheta^1 + \nabla_x \vartheta^0) - \mathcal{D}_s \operatorname{div}_x \underbrace{\nabla_y \vartheta^0}_{=0} = 0 \quad \text{in} \quad Y_s, \tag{98}$$

$$\mathcal{D}_f(\nabla_y \vartheta^1 + \nabla_x \vartheta^0) \cdot \mathbf{n} = \mathcal{D}_s(\nabla_y \vartheta^1 + \nabla_x \vartheta^0) \cdot \mathbf{n} \quad \text{and} \quad [\vartheta^1] = 0 \quad \text{on} \quad S, \tag{99}$$

$$\vartheta^1 \quad \text{is} \quad 1 - \text{periodic}, \tag{100}$$

for every $(x,t) \in \Omega \times (0, \mathcal{T})$. Fredholm's alternative implies that problem (89) has a solution. This solution is unique if we impose $\int_Y \vartheta^1 \, dy = 0$.

For ϑ^1 a separation of scales is possible, which is achieved through the temperature auxiliary problems, for $j = 1, 2, 3$,

$$-\mathcal{D}_f \operatorname{div}_y(\nabla_y \Theta_j + \mathbf{e}_j) = 0 \quad \text{in} \quad Y_f, \tag{101}$$

$$-\mathcal{D}_s \operatorname{div}_y(\nabla_y \Theta_j + \mathbf{e}_j) = 0 \quad \text{in} \quad Y_s, \tag{102}$$

$$\mathcal{D}_f(\nabla_y \Theta_j + \mathbf{e}_j) \cdot \mathbf{n} = \mathcal{D}_s(\nabla_y \Theta_j + \mathbf{e}_j) \cdot \mathbf{n} \quad \text{and} \quad [\Theta_j] = 0 \quad \text{on} \quad S, \tag{103}$$

$$\Theta_j \quad \text{is} \quad 1 - \text{periodic}, \quad \int_Y \Theta_j \, dy = 0. \tag{104}$$

Then ϑ^1 is given by the scale separation formula

$$\vartheta^1(x, y, t) = \sum_{j=1}^{3} \Theta_j(y) \frac{\partial \vartheta^0}{\partial x_j}(x, t). \tag{105}$$

In order to have an equation for ϑ^0, we need the next order terms in the expansions (90)-(92). They yield the following problem at order $O(1)$:

$$-\mathcal{D}_f \operatorname{div}_y(\nabla_y \vartheta^2 + \nabla_x \vartheta^1) + C_f \frac{\partial \vartheta^0}{\partial t} + C_f(\mathbf{w}^1 \cdot \underbrace{\nabla_y \vartheta^0}_{=0} +$$

$$\mathbf{w}^0 \cdot (\nabla_y \vartheta^1 + \nabla_x \vartheta^0)) - \mathcal{D}_f \operatorname{div}_x \nabla_y \vartheta^1 - \mathcal{D}_f \Delta_x \vartheta^0 = \tilde{h}_f \quad \text{in} \quad Y_f, \tag{106}$$

$$-\mathcal{D}_s \operatorname{div}_y(\nabla_y \vartheta^2 + \nabla_x \vartheta^1) - \mathcal{D}_s \operatorname{div}_x \nabla_y \vartheta^1 - \mathcal{D}_s \Delta_x \vartheta^0 + C_s \frac{\partial \vartheta^0}{\partial t} +$$

$$K_T \frac{\partial}{\partial t}(\operatorname{div}_y \mathbf{u}^1 + \operatorname{div}_x \mathbf{u}^0) = \tilde{h}_s \quad \text{in} \quad Y_s, \tag{107}$$

$$\mathcal{D}_f(\nabla_y \vartheta^2 + \nabla_x \vartheta^1) \cdot \mathbf{n} = \mathcal{D}_s(\nabla_y \vartheta^2 + \nabla_x \vartheta^1) \cdot \mathbf{n} \quad \text{and} \quad [\vartheta^2] = 0 \quad \text{on} \quad S, \tag{108}$$

$$\vartheta^2 \quad \text{is} \quad 1 - \text{periodic}, \tag{109}$$

for every $(x,t) \in \Omega \times (0, \mathcal{T})$. By Fredholm's alternative, problem (106)-(109) admits a solution if

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and only if

$$K_T \frac{\partial}{\partial t}\left(\int_{Y_s} \operatorname{div}_y \mathbf{u}^1 \, dy + (1-\Phi) \operatorname{div}_x \mathbf{u}^0\right) + \underbrace{(C_f \Phi + C_s(1-\Phi))}_{=1} \frac{\partial \vartheta^0}{\partial t} + C_f \int_{Y_f} \mathbf{w}^0 \cdot (\nabla_y \vartheta^1 + \nabla_x \vartheta^0) \, dy -$$

$$\operatorname{div}_x\left(\mathcal{D}_f \int_{Y_f} \nabla_y \vartheta^1 \, dy + \mathcal{D}_s \int_{Y_s} \nabla_y \vartheta^1 \, dy + (\mathcal{D}_f \Phi + \mathcal{D}_s(1-\Phi))\nabla_x \vartheta^0\right) = \tilde{h}_f \Phi + \tilde{h}_s(1-\Phi). \quad (110)$$

Equation (110) can be useful only if we are able to eliminate ϑ^1. Substituting (105) into the fourth term (divergence of the effective heat flux) of (110) yields

$$\mathcal{D}_f \int_{Y_f} \nabla_y \vartheta^1 \, dy + \mathcal{D}_s \int_{Y_s} \nabla_y \vartheta^1 \, dy + (\mathcal{D}_f \Phi + \mathcal{D}_s(1-\Phi))\nabla_x \vartheta^0 = \sum_{j=1}^{3}(\mathcal{D}_f \int_{Y_f} \nabla_y \Theta_j \, dy +$$

$$\mathcal{D}_s \int_{Y_s} \nabla_y \Theta_j \, dy)\frac{\partial \vartheta^0}{\partial x_j}(x,t) + (\mathcal{D}_f \Phi + \mathcal{D}_s(1-\Phi))\nabla_x \vartheta^0 =: \mathcal{B}\nabla_x \vartheta^0,$$

where

$$\mathcal{B}_{ij} = \mathcal{D}_f \int_{Y_f} \frac{\partial}{\partial y_i}\Theta_j \, dy + \mathcal{D}_s \int_{Y_s} \frac{\partial}{\partial y_i}\Theta_j \, dy + (\mathcal{D}_f \Phi + \mathcal{D}_s(1-\Phi))\delta_{ij}, \quad i,j = 1,2,3. \quad (111)$$

$\mathcal{B}$ is the effective heat dispersivity. It is easy to see that it is a symmetric positive definite matrix (see for instance [31] or [25]).

We also eliminate ϑ^1 from the transport term in (110). This gives

$$\int_{Y_f} \mathbf{w}^0 \cdot (\nabla_y \vartheta^1 + \nabla_x \vartheta^0) \, dy = (\int_{Y_f} \mathbf{w}^0 \, dy) \cdot \nabla_x \vartheta^0 + \int_{Y_f} \mathbf{w}^0 \cdot \nabla_y \vartheta^1 \, dy = (\int_{Y_f} \mathbf{w}^0 \, dy) \cdot \nabla_x \vartheta^0 -$$

$$\int_{Y_f} \underbrace{\operatorname{div}_y \mathbf{w}^0}_{=0} \vartheta^1 \, dy + \int_{S} \underbrace{\mathbf{w}^0 \cdot \mathbf{n}}_{=0} \vartheta^1 \, dS_y = (\int_{Y_f} \mathbf{w}^0 \, dy) \cdot \nabla_x \vartheta^0.$$

Hence equation (110) takes the form

$$K_T \frac{\partial}{\partial t}\left(\int_{Y_s} \operatorname{div}_y \mathbf{u}^1 \, dy + (1-\Phi) \operatorname{div}_x \mathbf{u}^0\right) + \frac{\partial \vartheta^0}{\partial t} + C_f (\int_{Y_f} \mathbf{w}^0 \, dy) \cdot \nabla_x \vartheta^0 -$$

$$\operatorname{div}_x\left(\mathcal{B}\nabla_x \vartheta^0\right) = \tilde{h}_f \Phi + \tilde{h}_s(1-\Phi). \quad (112)$$

After evaluating the average over Y_s of $\operatorname{div}_y \mathbf{u}^1$, it will represent the **homogenized equation for the temperature**.

STEP 5

Now we return to equation (89) for the solid displacement. Since this problem is linear, we decompose $\mathbf{u}^1$ through the scale separation formula

$$\mathbf{u}^1(x,y,t) = \sum_{i,j=1}^{3} (e_x(\mathbf{u}^0(x,t)))_{ij}\mathbf{a}^{ij}(y) + (p^0(x,t) - p_h x_3 - K_S \vartheta^0(x,t))\mathbf{b}^0(y), \quad (113)$$

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Here the displacement fields $\mathbf{a}^{ij}$ and $\mathbf{b}^0$ depend only on the geometry and are defined through the auxiliary problems for $i, j = 1, 2, 3$:

$$\begin{cases} \operatorname{div}_y\big(A(e_y(\mathbf{a}^{ij}) + \frac{1}{2}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i))\big) = 0 \text{ in } Y_s, \\ A(e_y(\mathbf{a}^{ij}) + \frac{1}{2}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i)) \cdot \mathbf{n} = 0 \text{ on } S, \\ \mathbf{a}^{ij} \quad \text{is 1-periodic} \end{cases} \tag{114}$$

and

$$\begin{cases} \operatorname{div}_y\big(A(e_y(\mathbf{b}^0))\big) = 0 \text{ in } Y_s, \\ Ae_y(\mathbf{b}^0) \cdot \mathbf{n} = -\mathbf{n} \text{ on } S, \quad \mathbf{b}^0 \text{ is 1-periodic.} \end{cases} \tag{115}$$

With the auxiliary problems defining $\mathbf{u}^1$ we associate the linear poroelasticity effective coefficients. They are given by the Gassman tensor

$$\mathcal{G}_{klij} = \Big(\int_{Y_s} A(\frac{1}{2}(\mathbf{e}^i \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^i) + e_y(\mathbf{a}^{ij})) \, dy \Big)_{kl}, \quad i, j, k, l = 1, 2, 3, \tag{116}$$

and the Biot matrix

$$B^H = \int_{Y_s} Ae_y(\mathbf{b}^0) \, dy, \tag{117}$$

with $\mathbf{a}^{ij}$ and $\mathbf{b}^0$ defined by the problems (114) and (115), respectively. The tensor $\mathcal{G}$ and the matrix B^H are symmetric and $\mathcal{G}$ is positive definite, see ([31], p. 129-190) .

Next we go back to the equations for the displacement (76), (88). More precisely we consider the terms of order $O(1)$ in (76) and of order $O(\varepsilon)$ in (77) and (80). This results in the following boundary value problem for $\mathbf{u}^2$

$$- \operatorname{div}_y(Ae_y(\mathbf{u}^2) + Ae_x(\mathbf{u}^1) - K_S \vartheta^1 I) - \operatorname{div}_x(Ae_y(\mathbf{u}^1) + Ae_x(\mathbf{u}^0) - K_S \vartheta^0 I) =$$
$$-K_g \mathbf{e}_3 \text{ in } Y_s \times \Omega \times (0, \mathcal{T}), \tag{118}$$
$$-(A(e_y(\mathbf{u}^2) + e_x(\mathbf{u}^1)) - K_S \vartheta^1 I) \cdot \mathbf{n} = p^1 \mathbf{n} - 2\eta(\vartheta^0)e_y(\mathbf{w}^0) \cdot \mathbf{n} \quad \text{on } S \times \Omega \times (0, \mathcal{T}), \tag{119}$$
$$\mathbf{u}^2 \quad \text{is 1-periodic in y.} \tag{120}$$

Fredholm's alternative gives the compatibility condition

$$\operatorname{div}_x\{(1 - \Phi)(Ae_x(\mathbf{u}^0) - K_S \vartheta^0 I) + \int_{Y_s} Ae_y(\mathbf{u}^1) \, dy\} - K_g(1 - \Phi)\mathbf{e}_3 = \int_S (p^1 I - 2\eta(\vartheta^0)e_y(\mathbf{w}^0)) \cdot \mathbf{n} \, dS_y.$$

Since

$$\int_S (p^1 I - 2\eta(\vartheta^0)e_y(\mathbf{w}^0)) \cdot \mathbf{n} \, dS_y = - \int_{Y_f} (\nabla_y p^1 - 2 \operatorname{div}_y(\eta(\vartheta^0)e_y(\mathbf{w}^0))) \, dy,$$

(note that the normal vector $\mathbf{n}$ points from the solid part to the fluid) and using equation (84) yields

$$\operatorname{div}_x\{(1 - \Phi)(Ae_x(\mathbf{u}^0) - K_S \vartheta^0 I) + \int_{Y_s} Ae_y(\mathbf{u}^1) \, dy\} - K_g(1 - \Phi)\mathbf{e}_3 = \int_{Y_f} (-K_B \vartheta^0 \mathbf{e}_3 + \nabla_x p^0) \, dy. \tag{121}$$

To use this condition we have to evaluate the integral over Y_s. Here we use the decomposition formula (113):

$$\int_{Y_s} \{A(e_y(\mathbf{u}^1) + e_x(\mathbf{u}^0)) - K_S \vartheta^0 I\} \, dy = \sum_{i,j=1}^{3} e_x(\mathbf{u}^0(x,t)))_{ij} A(\mathbf{a}^{ij}(y)) +$$

$$\frac{1}{2}(\mathbf{e}^i \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^i) \, dy + (p^0(x,t) - p_h x_3) \int_{Y_s} A\mathbf{b}^0(y) - K_S \vartheta^0(x,t) \int_{Y_s} (A\mathbf{b}^0(y) + I) \, dy =$$

$$\mathcal{G}e_x(\mathbf{u}^0) + (p^0 - p_h x_3)B^H - K_S \vartheta^0 B^H - K_S |Y_s|\vartheta^0 I. \tag{122}$$

Writing equation (121) in the form

$$\text{div}_x \{ \int_{Y_s} (Ae_x(\mathbf{u}^0) + Ae_y(\mathbf{u}^1) - K_S \vartheta^0 I) \, dy - \Phi p^0 I \} = -K_B \Phi \vartheta^0 \mathbf{e}_3 + K_g(1 - \Phi)\mathbf{e}_3. \tag{123}$$

and substituting (122) into (123) gives

$$\text{div}_x \{ \mathcal{G}e_x(\mathbf{u}^0) + (p^0 - p_h x_3)B^H - K_S \vartheta^0 B^H - K_S |Y_s|\vartheta^0 I - \Phi p^0 I \} = -K_B \Phi \vartheta^0 \mathbf{e}_3 + K_g(1 - \Phi)\mathbf{e}_3.$$

Hence we have obtained the system of partial differential equations for the **effective solid displacement** $\mathbf{u}^0$:

$$-\text{div}_x \{ \mathcal{G}e_x(\mathbf{u}^0) \} + \text{div}_x \{ (\Phi I - B^H)(p^0 - p_h x_3) + K_S \vartheta^0(B^H + (1 - \Phi)I) \} =$$
$$K_B \Phi \vartheta^0 \mathbf{e}_3 - p_h \Phi \mathbf{e}_3 - K_g(1 - \Phi)\mathbf{e}_3. \tag{124}$$

$\boxed{\text{STEP 6}}$

Now we return to the problem (84)-(87). Since p^0 and ϑ^0 do not depend on y, we can use the scale separation from the classical derivation of the Darcy law via homogenization (see e.g. [2] or [31]). It uses the well known auxiliary problems in the homogenization of the filtration through rigid porous media and gives

$$\mathbf{w}^0(x,y,t) = \sum_{j=1}^{3} \frac{\mathbf{q}^j(y)}{\eta(\vartheta^0)} (K_B \vartheta^0 \delta_{j3} - \frac{\partial p^0}{\partial x_j}(x,t)), \tag{125}$$

$$p^1(x,y,t) = \sum_{j=1}^{3} \pi^j(y)(K_B \vartheta^0 \delta_{j3} - \frac{\partial p^0}{\partial x_j}(x,t)), \tag{126}$$

where $\{\mathbf{q}^j, \pi^j\}$ satisfy

$$-\Delta_y \mathbf{q}^j + \nabla_y \pi^j = \mathbf{e}_j \quad \text{and div } _y\mathbf{q}^j = 0 \quad \text{in } Y_f, \tag{127}$$
$$\mathbf{q}^j = 0 \quad \text{on } S, \quad \{\mathbf{q}^j, \pi^j\} \quad \text{is 1-periodic in y.} \tag{128}$$

Then

$$\mathcal{K}_{ij} := \int_{Y_f} q_j^i(y) \, dy \tag{129}$$

is the classical permeability tensor in Darcy's law, which is symmetric and positive definite (see for instance [31]).

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Consequently, the Darcy law reads

$$\mathbf{v}^D = \int_{Y_f} \mathbf{w}^0 \; dy = -\frac{\mathcal{K}}{\eta(\vartheta^0)}\big(\nabla_x p + (p_h - K_B\vartheta)\mathbf{e}_3\big).$$

It remains to find an effective equation for the pressure p^0.

We start with the equation at order $O(1)$ in the expansion of mass conservation condition (83). It reads

$$\text{div}_x\big(\mathbf{w}^0 + \frac{\tau_T}{\tau_c}\frac{\partial \mathbf{u}^0}{\partial t}\big) + \; \text{div}_y\big(\mathbf{w}^1 + \frac{\tau_T}{\tau_c}\frac{\partial \mathbf{u}^1}{\partial t}\big) = 0.$$

Integration of this equation in Y_f yields

$$\text{div}_x\big(\int_{Y_f} \mathbf{w}^0 \; dy + \Phi\frac{\tau_T}{\tau_c}\frac{\partial \mathbf{u}^0}{\partial t}\big) = -\frac{\tau_T}{\tau_c}\int_{Y_f} \text{div}_y\frac{\partial \mathbf{u}^1}{\partial t} \; dy =$$

$$\frac{\tau_T}{\tau_c}\partial_t\int_{Y_s} \text{div}_y\mathbf{u}^1(x,y,t) \; dy. \tag{130}$$

Recalling, see [24],

$$B_{ij}^H = \int_{Y_s} (Ae_y(\mathbf{b}^0)(y))_{ij} \; dy = \frac{1}{2}\int_{Y_s} Ae_y(\mathbf{b}^0)(y) : (\mathbf{e}^i \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^i) \; dy$$

$$= -\int_{Y_s} Ae_y(\mathbf{b}^0)(y) : e_y(\mathbf{a}^{ij}) \; dy = \int_{Y_s} \text{div}_y\mathbf{a}^{ij}(y) \; dy, \tag{131}$$

introducing the coefficient, see again [24]

$$\frac{1}{M} := -\int_{Y_s} \text{div}_y\mathbf{b^0} \; dy = \int_{Y_s} Ae_y(\mathbf{b}^0) : e_y(\mathbf{b}^0) \; dy > 0 \tag{132}$$

and evaluating

$$\int_{Y_s} \text{div}_y\mathbf{u}^1(x,y,t) \; dy = \sum_{i,j=1}^{3} (e_x(\mathbf{u}^0(x,t)))_{ij}\int_{Y_s} \text{div}_y\mathbf{a}^{ij}(y) \; dy+$$

$$(p^0(x,t) - p_h x_3 - K_S\vartheta^0(x,t))\int_{Y_s} \text{div}_y\mathbf{b}^0(y) \; dy, \tag{133}$$

we replace $\int_{Y_s} \text{div}_y\mathbf{a}^{ij}(y) \; dy$ by B_{ij}^H and $\int_{Y_s} \text{div}_y\mathbf{b}^0(y) \; dy$ by $-1/M$. After inserting (133) into (130), we find the the pressure equation of the quasi-static thermal Biot model:

$$\partial_t\Big(\frac{1}{M}p^0 - \frac{K_S}{M}\vartheta^0 + (\Phi I - B^H) : e_x(\mathbf{u}^0)\Big) + \frac{\tau_c}{\tau_T} \; \text{div}_x\big(\frac{\mathcal{K}}{\eta(\vartheta^0)}(K_B\vartheta^0\mathbf{e}_3 - \nabla_x p^0)\big) = 0. \tag{134}$$

STEP 7

After the computation from the previous step, we are able to write more explicitly the temperature equation. We substitute (133) into (112) to finally obtain:

$$K_T\frac{\partial}{\partial t}\big(-\frac{1}{M}(p^0 - p_h x_3) + (B^H + (1-\Phi)I) : e_x(\mathbf{u}^0)\big) + (1 + \frac{K_S K_T}{M})\frac{\partial\vartheta^0}{\partial t}+$$

$$C_f(\int_{Y_f} \mathbf{w}^0 \; dy)\cdot\nabla_x\vartheta^0 - \; \text{div}_x\Big(\mathcal{B}\nabla_x\vartheta^0\Big) = \tilde{h}_f\Phi + \tilde{h}_s(1-\Phi). \tag{135}$$

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Remark 8. *(Non-isotropic media): In case of non-isotropic tensors A and $\mathcal{M}$ one uses equations (22) and (23) as starting point for the two-scale expansions. Then the upscaled temperature equation (112) is replaced by the expression*

$$K_T \frac{\partial}{\partial t}\Big(\int_{Y_s} \mathcal{M} : e_y(\mathbf{u}^1)\ dy + (1-\Phi)\mathcal{M} : e_x(\mathbf{u}^0)\Big) + \frac{\partial \vartheta^0}{\partial t} + C_f\Big(\int_{Y_f} \mathbf{w}^0\ dy\Big) \cdot \nabla_x \vartheta^0 -$$

$$\mathrm{div}_x\Big(\mathcal{B}\nabla_x\vartheta^0\Big) = \tilde{h}_f\Phi + \tilde{h}_s(1-\Phi). \tag{136}$$

The scale separation formula for $\mathbf{u}^1$ becomes more involved. An additional auxiliary problem of the form

$$\begin{cases} \mathrm{div}_y\big(A(e_y(\mathbf{b}^1)\big) = 0\ in\ Y_s, \\ Ae_y(\mathbf{b}^1) \cdot \mathbf{n} = \mathcal{M}\mathbf{n}\ on\ S, \quad \mathbf{b}^1\ is\ 1\text{-periodic}, \end{cases} \tag{137}$$

is needed. This is used in the separation formula which now reads

$$\mathbf{u}^1(x,y,t) = \sum_{i,j=1}^{3} (e_x(\mathbf{u}^0(x,t)))_{ij}\mathbf{a}^{ij}(y) + (p^0(x,t) - p_h x_3)\mathbf{b}^0(y) + K_S\vartheta^0(x,t)\mathbf{b}^1(y). \tag{138}$$

Then arguing as is Step 5, we obtain the modified (non-isotropic) Biot equation

$$-\,\mathrm{div}_x\{\mathcal{G}e_x(\mathbf{u}^0)\} + \,\mathrm{div}_x\{(\Phi I - B^H)(p^0 - p_h x_3) - K_S\vartheta^0(\mathcal{M}^H - (1-\Phi)\mathcal{M})\} =$$

$$K_B\Phi\vartheta^0\mathbf{e}_3 - p_h\Phi\mathbf{e}_3 - K_g(1-\Phi)\mathbf{e}_3, \tag{139}$$

where $\mathcal{M}^H = \int_{Y_s} Ae_y(\mathbf{b}^1)\ dy$. Similarly, the mass conservation equation (134) changes into

$$\partial_t\Big(\frac{p^0}{M} + K_S\vartheta^0\big(\int_{Y_s} \mathrm{div}_y\mathbf{b}^1(y)\ dy\big) + (\Phi I - B^H) : e_x(\mathbf{u}^0)\Big) +$$

$$\frac{\tau_c}{\tau_T}\,\mathrm{div}_x\Big(\frac{\mathcal{K}}{\eta(\vartheta^0)}(K_B\vartheta^0\mathbf{e}_3 - \nabla_x p^0)\Big) = 0. \tag{140}$$

Finally, the modified temperature equation is obtained by substituting expression (138) into equation (136). Details are omitted.

4 The upscaled model and a corresponding Lyapunov functional

Let the *Darcy velocity* (i.e. the *upscaled fluid velocity relative to the solid skeleton*) be given by

$$\mathbf{v}^D = \int_{Y_f} \mathbf{w}^0\ dy. \tag{141}$$

With this definition the homogenization procedure from Section 3 gives the following set of equations (for $(x,t) \in \Omega \times (0, \mathcal{T})$) :

    Darcy's law

$$\mathbf{v}^D = -\frac{\mathcal{K}}{\eta(\vartheta^0)}\big(\nabla_x p + (p_h - K_B\vartheta)\mathbf{e}_3\big), \tag{142}$$

where p denotes the dimensionless actual fluid pressure (i.e. including the hydrostatic component) and $p_h - K_B\vartheta$ is proportional to the dimensionless density of the fluid phase.

Momentum balance

$$-\operatorname{div}_x\sigma = K_B\Phi\vartheta\mathbf{e}_3 - p_h\Phi\mathbf{e}_3 - K_g(1-\Phi)\mathbf{e}_3, \tag{143}$$

where the *total stress* σ is given by

$$\sigma = \mathcal{G}e_x(\mathbf{u}) - (\Phi I - B^H)p - K_S\vartheta(B^H + (1-\Phi)I). \tag{144}$$

Mass balance

$$\partial_t\zeta + \frac{\tau_c}{\tau_T}\operatorname{div}_x\mathbf{v}^D = 0, \tag{145}$$

where the *change in volume fluid content* ζ is given by

$$\zeta = \frac{p - K_S\vartheta}{M} + (\Phi I - B^H) : e_x(\mathbf{u}); \tag{146}$$

Energy balance

$$K_T\frac{\partial}{\partial t}\big(-\zeta + \operatorname{div}_x\mathbf{u}\big) + \frac{\partial\vartheta}{\partial t} + C_f\mathbf{v}^D\cdot\nabla_x\vartheta - \operatorname{div}_x\Big(\mathcal{B}\nabla_x\vartheta\Big) = \tilde{h}_f\Phi + \tilde{h}_s(1-\Phi). \tag{147}$$

Note that the upscaled system (142)-(147) preserves the basic conservation law properties of the micro-scale system. Furthermore, it has the following important stability property:

Proposition.
Let

$$2W(e, p, \vartheta) = \mathcal{G}e : e + \frac{1}{M}(p - K_S\vartheta)^2 + \frac{K_S}{K_T}\vartheta^2, \tag{148}$$

where e is taking values in the space of symmetric square matrices on $\mathbb{R}^3$ and p and ϑ are scalars. Then, in the absence of the gravity and other external forces and for small ΔT, W is a Lyapunov functional for the solutions of the system (142)-(147), i.e.

$$\int_\Omega W(e_x(\mathbf{u}), p, \vartheta)(t)\ dx$$

is monotone decreasing in time.

Sketch of the proof. For simplicity we suppose homogeneous Dirichlet's conditions for $\mathbf{u}$ and homogeneous Neumann conditions for p and ϑ. First we multiply equation (143) by $\partial_t\mathbf{u}$ and integrate over Ω. This yields

$$\int_\Omega \mathcal{G}e_x(\mathbf{u}) : e_x(\partial_t\mathbf{u})\ dx - \int_\Omega (\Phi I - \mathcal{B}^H) : e_x(\partial_t\mathbf{u})p\ dx -$$
$$\int_\Omega ((1-\Phi)I + \mathcal{B}^H) : e_x(\partial_t\mathbf{u})K_S\vartheta\ dx = 0. \tag{149}$$

Next we test equation (145) by p and obtain

$$\int_\Omega \frac{p - K_S \vartheta}{M} \partial_t p \; dx + \int_\Omega \left(\Phi I - \mathcal{B}^H\right) : e_x(\partial_t \mathbf{u}) p \; dx - \int_\Omega \mathbf{v}^D \cdot \nabla_x p \; dx = 0. \tag{150}$$

Finally, equation (147) is multiplied by K_S/K_T and then tested by ϑ. This gives

$$-K_S \int_\Omega \partial_t \frac{p - K_S \vartheta}{M} \vartheta \; dx + K_S \int_\Omega (B^H + (1 - \Phi)I) : e_x(\partial_t \mathbf{u}) \vartheta \; dx + \frac{K_S}{K_T} \int_\Omega \partial_t \vartheta \vartheta \; dx +$$
$$\frac{K_S}{K_T} \int_\Omega C_f \mathbf{v}^D \cdot \nabla_x \vartheta \vartheta \; dx + \frac{K_S}{K_T} \int_\Omega \mathcal{B} \nabla_x \vartheta \cdot \nabla_x \vartheta \; dx = 0. \tag{151}$$

Summing up the above three equations, we observe that most of the cross terms cancel and there remains

$$\frac{1}{2} \partial_t \int_\Omega \{ \mathcal{G} e_x(\mathbf{u}) : e_x(\mathbf{u}) + \frac{p^2}{M} - \frac{2K_S p \vartheta}{M} + (\frac{K_S^2}{M} + \frac{K_S}{K_T}) \vartheta^2 \} \; dx +$$
$$\frac{K_S}{K_T} \int_\Omega \left(\mathcal{B} \nabla_x \vartheta \cdot \nabla_x \vartheta - C_f \frac{\mathcal{K}}{\eta(\vartheta)} \nabla_x p \cdot \nabla_x \vartheta \vartheta + \frac{\tau_c K_T}{\tau_T K_S} \frac{\mathcal{K}}{\eta(\vartheta)} \nabla_x p \cdot \nabla_x p \right) \; dx = 0. \tag{152}$$

Since

$$\frac{p^2}{M} - \frac{2K_S p \vartheta}{M} + (\frac{K_S^2}{M} + \frac{K_S}{K_T}) \vartheta^2 = \frac{(p - K_S \vartheta)^2}{M} + \frac{K_S}{K_T} \vartheta^2,$$

one can write (152) as

$$\frac{1}{2} \frac{d}{dt} \int_\Omega W(e_x(\mathbf{u}), p, \vartheta) \; dx +$$
$$\frac{K_S}{K_T} \int_\Omega \left(\mathcal{B} \nabla_x \vartheta \cdot \nabla_x \vartheta - C_f \vartheta \frac{\mathcal{K}}{\eta(\vartheta)} \nabla_x p \cdot \nabla_x \vartheta + \frac{\tau_c K_T}{\tau_T K_S} \frac{\mathcal{K}}{\eta(\vartheta)} \nabla_x p \cdot \nabla_x p \right) \; dx = 0. \tag{153}$$

It remains to study the influence of the terms, containing the gradients. Observe that these terms are a quadratic form in $\nabla_x \vartheta$ and $\nabla_x p$. Manipulating this quadratic form, one observes that a sufficient condition for its nonnegativity is the absence of real roots for the polynomial $\xi \to \xi^2 - C_f \|\mathcal{B}\mathcal{K}^{-1}\vartheta\|_\infty^{1/2} \frac{\xi}{\eta_{min}} + \frac{\tau_c K_T}{\tau_T K_S \eta_{max}}$. Here η_{min} and η_{max} are, respectively, the minimal and maximal values of $\eta(\vartheta)$. Since $\frac{\tau_c K_T}{\tau_T K_S} = \frac{\Lambda T_0 u_c}{\tilde{\rho} c L (\Delta T)^2}$, the nonnegativity holds for ΔT sufficiently small.Text $\qquad \square$

Remark 9. *Note that the ratio K_S/K_T is fundamental in both the definition of the free energy and in the control of the dissipation.*

In [12], a free energy is introduced which is a function of the variables e, ζ and ϑ. Rewriting (148) in terms of these variables gives

$$2W(e, \zeta, \vartheta) = \mathcal{G} e : e + M(\zeta - (\Phi I - B^H) : e)^2 + \frac{K_S}{K_T} \vartheta^2.$$

This expression reduces exactly to Biot's free energy [8] in the isothermal case, where $\vartheta = 0$, and it has qualitatively the same form as the reduced potential in [12], page 4634. Note that $\frac{\partial}{\partial e}(W(e, \zeta, \vartheta) + K_S \vartheta(\zeta - I : e)) = \sigma$ and $\frac{\partial}{\partial \zeta}(W(e, \zeta, \vartheta) + K_S \vartheta(\zeta - I : e)) = p$.

Remark 10. *(Comparison to literature) Recasting expressions (144) and (146) into the form*

$$\sigma = \mathcal{G}e_x(\mathbf{u}) + M(\Phi I - B^H)(\Phi I - B^H) : e_x(\mathbf{u}) - M(\Phi I - B^H)\zeta - K_S \vartheta I \tag{154}$$

and

$$p = M\zeta + K_S\vartheta - (\Phi I - B^H) : e_x(\mathbf{u}), \tag{155}$$

we are able to make a comparison with the consitutive laws (106)-(107) from [12] (page 4634). We find that both σ and p contain similar terms, of which now the matricial coefficients are computable in terms of the microstructure.

Comparing with [19], the constitutive law for the total stress σ is in agreement with equation (144) in its elastic and pressure parts, but not in the temperature part. In [19] an additional auxiliary cell problem for the thermal coefficient was introduced. It turns out that the result is identical to our matricial coefficient for the temperature in (144). Concerning the constitutive equation (146) for the volume content ζ, we observe that in [19] (equation (126), page 135) the expression contains in addition a term involving the gradient of porosity. Furthermore, the temperature equation in [19] (equation (143), page 137) has the transport velocity that contains the effective thermal diffusivity.

For future reference we present the upscaled model in dimensional form. Then it reads

$$\frac{\partial}{\partial t}\left\{\frac{P}{\Lambda M} - \frac{4 - 2\nu}{(1 + \nu)(1 - 2\nu)}\frac{\beta_s T}{M} + \text{div } _x\big((\Phi I - \mathcal{B}^H)\mathbf{U}\big)\right\} + \text{div}_x\mathbf{V}^D = 0, \tag{156}$$

$$\mathbf{V}^D = \frac{\mathcal{K}\ell^2}{\eta(T)}\big(\rho_f^0\beta_f g(T - T_0 - \frac{1}{\beta_f})\mathbf{e}_3 - \nabla_x P\big), \tag{157}$$

$$\big(\tilde{\rho}c + \frac{\beta_s^2\Lambda T_0}{M}\big)\frac{\partial T}{\partial t} + \Lambda\beta_s T_0\frac{\partial}{\partial t}\big(-\frac{P}{\Lambda M} - (\Phi I - \mathcal{B}^H) : e_x(\mathbf{U})\big) - $$
$$\text{div } _x\big(\mathcal{B}^{term}\nabla_x T\big) + \rho_f^0 c_f\mathbf{V}^D \cdot \nabla_x T = h_f\Phi + h_s(1 - \Phi), \tag{158}$$

$$- \text{div}_x\Big(\mathcal{G}\Lambda e_x(\mathbf{U}) - \big(\Phi I - \mathcal{B}^H\big)P - \big((1 - \Phi)I + \mathcal{B}^H\big)(3\lambda + 2\nu)\beta_s(T - T_0)\Big) = $$
$$\rho_f^0\beta_f\Phi g(T - T_0)\mathbf{e}_3 - (\rho_f^0\Phi + \rho_s(1 - \Phi))g\mathbf{e}_3, \tag{159}$$

where

$$\mathcal{B}^{therm} = \frac{\tilde{\rho}c}{\rho_f^0 c_f}\kappa_f \text{Pe}\mathcal{B} \quad \text{(the homogenized thermal dispersion coefficient)}.$$

All coefficients and variables appearing in (156)-(159) are defined in Table 3.

We conclude this section by two generalizations of the dimensionless upscaled model (141)-(147). We make them the content of the following two remarks.

Remark 11. *(Weak compressibility of the fluid)*
Allowing for weak fluid compressibility, the mass balance reads, see also [[24], eq. (2.3)],

$$\alpha\frac{\partial p}{\partial t} + \text{div}_x\mathbf{v}_f = 0,$$

where α denotes the fluid compressibility which depends on the fluid bulk modulus and density.

	QUANTITY	UNIT
$\mathbf{U} = u_c \mathbf{u}^0$	solid phase displacement	m (meter)
L	domain size	m (meter)
$p_h p_c = \rho_f^0 g L$	hydrostatic pressure	Pa (Pascal)
$P = p_c(p^0 - p_h x_3)$	fluid phase pressure	Pa (Pascal)
$T = \Delta T \vartheta^0 + T_0$	temperature	K (Kelvin)
$\eta = \eta(T) = \eta_c \eta(\vartheta^0)$	dynamic viscosity	$kg/(m\,s)$
ℓ	pore size	m (meter)
$\mathcal{B}^H$	part of Biot's parameter	a constant dimensionless matrix
$\mathcal{B}^{therm} = \mathcal{B}\frac{\bar{\rho}c}{\rho_f^0 c_f}\kappa_f \mathrm{Pe}$	homogenized thermal diffusivity coefficient	m^2/s
$\mathcal{K}^{phys} = \ell^2 \mathcal{K}$	permeability	m^2
Φ	porosity	dimensionless with values in $(0,1)$
$|Y_s| = 1 - \Phi$	solid phase volume part	
β	coefficient of thermal expansion	$1/K$
$K_S = \beta_s \Delta T L/u_c$	parameter containing thermal effects	dimensionless
Pe	Péclet's number	dimensionless
Λ	characteristic value of Young's moduli	Pa (Pascal)
$\Lambda \mathcal{G}$	Gassmann's 4th order positive definite symmetric tensor	Pa (Pascal)
$c_\rho \rho_f^0 c_f$	effective heat capacity	dimensionless
$h_f \bar{h}$	heat source	K/s
$K_g = \frac{L^2 \rho_s g}{\Lambda u_c}$	Bousinessq force	dimensionless
$K_B = C_0 K_S = K_S \frac{K_g \rho_f^0 \beta_f \Delta T}{K_S \rho_s}$	buoyancy	$O(K_S)$
$\Lambda_c M > 0$	compressibility	Pa

Table 3: *Description of parameters and unknowns.*

In dimensionless form we find

$$\kappa_c \frac{\partial p^\varepsilon}{\partial t} + \mathrm{div}(\mathbf{w}^\varepsilon + \frac{\tau_T}{\tau_c}\frac{\partial \mathbf{u}^\varepsilon}{\partial t}) = 0, \tag{160}$$

where $\kappa_c = \alpha p_c$.

This equation replaces (63) in Ω_f^ε and in the expansions.

With minor modifications we obtain the same upscaled equations, except equation (145) which now takes the form

$$\partial_t(\Phi \kappa_c p + \zeta) + \frac{\tau_c}{\tau_T}\,\mathrm{div}_x \mathbf{v}^D = 0. \tag{161}$$

Remark 12. *(Nonlinear heat capacity)*

The upscaled model (141)-(147) was derived under the assumption that the temperature T does not deviate much from the reference temperature T_0. Furthermore we introduced constant heat capacities for fluid and solid. We recall that the heat capacity is defined as $T\frac{\partial S}{\partial T}$. This quantity proves to be virtually independent of the temperature, see for instance [34].

However in our derivation we use the approximation $T_0 \frac{\partial S}{\partial T}$. Hence for large deviation from T_0, we need to correct the heat capacity by a factor $\frac{T_0}{T}$. To cover the whole range of T-values we introduce the temperature dependence

$$R(T) = \sqrt{\frac{T_0^2 + \delta^2}{T^2 + \delta^2}}, \tag{162}$$

where δ is small parameter. Note that $R(T) \approx 1$ for T close to T_0 and $R(T) \approx \dfrac{1}{T}$ for large T. In the dimensionless setting this leads to the heat capacities

$$C_i(\vartheta) = C_i R(\vartheta) \quad for \ \ i = f, s, \tag{163}$$

where

$$R(\vartheta) = \sqrt{\frac{1 + \dfrac{\delta^2}{T_0^2}}{1 + \dfrac{\delta^2}{T_0^2} + \dfrac{2\Delta T \vartheta}{T_0} + (\dfrac{\Delta T \vartheta}{T_0})^2}}, \quad 1 \gg \delta > 0, \ T_0 > 0, \ \Delta T > 0. \tag{164}$$

Note that also in the presence of nonlinear heat capacities the Boussinesq approximation remains valid, see e.g. [15].

Again, with minor modifications we obtain the same upscaled equations, except equation (147) which now reads

$$K_T \frac{\partial}{\partial t}\Big(-\zeta + \ \mathrm{div}_x \mathbf{u} \Big) + R(\vartheta) \frac{\partial \vartheta}{\partial t} + C_f R(\vartheta) \mathbf{v}^D \cdot \nabla_x \vartheta - \ \mathrm{div}_x \Big(\mathcal{B} \nabla_x \vartheta \Big) = \tilde{h}_f \Phi + \tilde{h}_s(1 - \Phi). \tag{165}$$

5 A pore network model: calculation of the homogenized coefficients

In this section, we compute the upscaled matricial coefficients that appear in the equations (142)-(147). Specifically, these are

- the thermal diffusion matrix $\mathcal{B}$ (111),

- the Gassman tensor $\mathcal{G}$ (116),

- the Biot matrix B^H (117),

- the coefficient $1/M$ (132),

- and the permeability tensor $\mathcal{K}$ (129).

They are obtained by solving the corresponding auxiliary cell problems

- Cell problem 1: temperature (101)-(104),

- Cell problem 2: solid in terms of elasticity for $\mathbf{a}^{ij}$ (114),

- Cell problem 3: solid in terms elasticity for $\mathbf{b}^0$ (115),

- Cell problem 4: fluid in terms of Stokes (127)-(128).

5.1 Discretization

The discretization of the four cell problems is based on a Galerkin finite element scheme. As usually done, the strong forms are integrated over the domain $Y = (0,1)^3$ to obtain variational formulations. The grid consists of hexahedral cells. We introduce H^1 conforming discrete spaces (see e.g., [10]), which consist of continuous trilinear functions Q_1^c for the thermal problem, and correspondingly vector-valued spaces for the two elasticity cell problems. The Stokes equations are discretized with the Taylor-Hood element in which triquadratic Q_2^c functions are adopted for the velocity and Q_1^c functions for the pressure, yielding an inf-sup stable finite element pair. The (linear) discrete systems can be solved with a conjugate gradient method for the thermal and elasticity cell problems and a Schur complement technique for the Stokes system. The basic programming code is based on the deal.II finite element library ([4], [6]) and uses parts of deal.II-step-22, deal.II-step-45, and the fluid-structure program [36].

5.2 Numerical simulations

We now provide more details on the geometry, boundary conditions and material parameters and finally discuss our numerical results.

Geometries. We consider two specific porous media that have the form of the pore network models from Figure 1. Figure 1 (left) shows the base case, in which the geometry is isotropic and Figure 1 (right) shows a non-isotropic case, where the cross section of the channels in (y, z)-directions is different from the cross section in the x-direction. With respect to the base case we take the cross section of the channels in (y, z)-directions the same and in x-direction a factor 2×2 larger. In the computations we impose 1-periodicity of solutions to the cell problems.

The dimension of the fluid part in the unit cell is such that

$$\Phi = |Y_f| = \begin{cases} 0.0820312 & \text{in the isotropic case;} \\ 0.111328 & \text{in the non-isotropic case.} \end{cases}$$

Material parameters. We suppose that the elastic material of the solid part is isotropic, with the elasticity tensor A given by (16). Since the coefficients are divided by the Young's modulus Λ, we just need to set the Poisson ratio. Here we take $\nu = 0.25$. This implies that the dimensionless Lamé first parameter λ and the shear modulus μ are both equal to 0.4 and that the corresponding dimensionless bulk modulus is $K = 2/3$. Since the ratio of $\mathcal{D}_s/\mathcal{D}_f = 10$ for both clay and rock, we run only one set of auxiliary problems where $\mathcal{D}_f = 0.1$ and $\mathcal{D}_s = 1$. The actual results for clay and rock are obtained by a trivial scaling.

Discussion of our findings. We discuss our findings in the previous order: temperature, elasticity, and Stokes.

Cell problem 1: Thermal diffusivity: The results are given in terms of $B = \mathcal{B}/\mathcal{D}_s$ (by scaling). For the two specific geometries considered here, we calculate for the isotropic case the following values of the thermal diffusion matrix:

$$B = \begin{bmatrix} 0.982703 & 6.47568e - 06 & 6.47568e - 06 \\ 6.47568e - 06 & 0.982703 & 6.47565e - 06 \\ 6.47568e - 06 & 6.47565e - 06 & 0.982703 \end{bmatrix}$$

and for the non-isotropic case, we obtain:

$$B = \begin{bmatrix} 0.950041 & 6.32571e-06 & 6.32571e-06 \\ 6.32571e-06 & 0.982624 & 6.11721e-06 \\ 6.32571e-06 & 6.11721e-06 & 0.982624 \end{bmatrix}$$

Note that the matrices are (almost) diagonal, as expected from the geometries. The geometrical non-isotropy has only a minor effect, as can be seen from the diagonal terms in the matrix. This is due to the relatively small differences in porosity and the limited contrast $\mathcal{D}_s/\mathcal{D}_f = 10$. Clearly, the effect of non-isotropy could be enhanced by taking other geometries.

Cell problem 2: Gassmann tensor The Gassmann's tensor is written in Voigt's representation as a 6×6 symmetric matrix, involving twenty-one independent coefficients. For the isotropic case, we obtain:

$$\mathcal{G} = \begin{bmatrix} 1.21187 & 0.428817 & 0.428817 & -4.69e-18 & -3.78e-18 & 7.0e-17 \\ 0.428817 & 1.21187 & 0.428817 & -1.94e-16 & -7.45e-18 & -9.92e-18 \\ 0.428817 & 0.428817 & 1.21187 & 1.33e-17 & -2.97e-17 & -1.36e-17 \\ -4.69e-18 & -1.94e-16 & 1.33e-17 & 0.408828 & -1.08e-16 & 6.85e-19 \\ -3.78e-18 & -7.45e-18 & -2.97e-17 & -1.08e-16 & 0.408828 & 8.57e-17 \\ 7.0e-17 & -9.92e-18 & -1.36e-17 & 6.85e-19 & 8.572e-17 & 0.408828 \end{bmatrix}$$

and for the non-isotropic case (an orthotropic effective material), we calculate:

$$\mathcal{G} = \begin{bmatrix} 1.15945 & 0.434037 & 0.434037 & 1.03e-17 & 3.5e-18 & -3.83e-17 \\ 0.434037 & 1.2435 & 0.45122 & -5.49e-16 & -1.11e-17 & -7.62e-18 \\ 0.434037 & 0.45122 & 1.2435 & 1.68e-17 & -1.31e-16 & 1.13e-17 \\ 1.03e-17 & -5.49e-16 & 1.68e-17 & 0.439163 & -1.3e-17 & -3.44e-18 \\ 3.50433e-18 & -1.11e-17 & -1.31e-16 & -1.3e-17 & 0.426031 & 3.49e-17 \\ -3.83e-17 & -7.62e-18 & 1.13e-17 & -3.44e-18 & 3.49e-17 & 0.426031 \end{bmatrix}$$

Cell problem 3: Biot matrix and coefficient $1/M$: Here, we compute Biot's matrix B^H and the coefficient $1/M$ again for the two specific geometries. The calculation for the isotropic case, yields:

$$B^H = \begin{bmatrix} 0.116782 & 2.93853e-17 & 1.12859e-16 \\ 2.93853e-17 & 0.116782 & -7.11201e-19 \\ 1.12859e-16 & -7.11201e-19 & 0.116782 \end{bmatrix}$$

and

$$\frac{1}{M} = 0.182911.$$

The computations for the non-isotropic case yield:

$$B^H = \begin{bmatrix} 0.125088 & 5.52671e-18 & 4.0376e-17 \\ 5.52671e-18 & 0.175705 & 7.36425e-17 \\ 4.0376e-17 & 7.36425e-17 & 0.175705 \end{bmatrix}$$

and

$$\frac{1}{M} = 0.255538.$$

As for the thermal diffusivity, the matrices are almost diagonal and again the effect of non-isotropy is minor. Note that the matrix $\Phi I - B^H \leq 0$ in both the isotropic and non-isotropic case.

Cell problem 4: Permeability: Solving the Stokes problem, we obtain for the isotropic case the following permeabilities:

$$\mathcal{K} = \begin{bmatrix} 1.24019e - 05 & -5.26e - 22 & -2.05e - 23 \\ -1.14e - 21 & 1.24019e - 05 & -3.21e - 21 \\ 3.18e - 23 & -4.14e - 21 & 1.24019e - 05 \end{bmatrix}$$

and for the non-isotropic case:

$$\mathcal{K} = \begin{bmatrix} 1.77062e - 04 & -1.38e - 22 & -3.70e - 20 \\ 2.80e - 21 & 1.24036e - 05 & 8.85e - 20 \\ 1.98e - 20 & -1.54e - 19 & 1.24036e - 05 \end{bmatrix}$$

Note that for the permeability the effect of anisotropy is well noticeable (as expected).

We finally remark that we have developed our codes for sequential programming and uniform grid refinement. Therefore, in these 3D simulations, the discretization error may be relatively large. For this paper, the obtained simulation values provide a very good understanding and most importantly substantiate our theory. To enhance the precision (i.e., reducing the discretization error) future studies may address local mesh adaptivity and parallel computing in which the last two authors have recent work (e.g., [21]) on very related problems in fractured porous media.

6 Conclusion

In this paper we derive the thermal Biot equations by the method of two-scale expansions. Starting at the pore level with the momentum and the energy balance equations in the fluid and in the solid phase, together with the appropriate conditions at the fluid-solid interface. We derive an upscaled formulation. It is given in terms of Darcy's law, and the balances for momentum, mass and energy. We also construct a Lyapunov functional for the upscaled system, which implies stable behavior of the corresponding solutions. We also provide details when comparing our results to those obtained by Coussy et al [12] and Lee and Mei [19].

Finally we consider, as an example, a periodic porous structure in the form of a pore network model. We evaluate numerically the matricial coefficients in the upscaled equations by solving the corresponding auxiliary problems. Two specific cases are considered: an isotropic structure and a non-isotropic structure, in which there is a preference in the single direction. The numerical results clearly demonstrate the effect of non-isotropy of the porous structure on the coefficients of the upscaled model.

This is part 1 of a series of papers in preparation on the subject. Future work focuses on rigorous mathematical analysis of the equations, on the analysis of benchmark problems and on numerical simulations of concrete nuclear waste and CCS examples.

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